Radiative Corrections to Electron-Proton Scattering

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The radiative corrections to elastic electron-proton scattering are analyzed in a hadronic model including the finite size of the nucleon. For initial electron energies above 8 GeV and large scattering angles, the proton vertex correction in this model increases by at least two percent the overall factor by which the one-photon exchange (Rosenbluth) cross section must be multiplied. The contribution of soft photon emission is calculated exactly. Comparison is made with the generally used expressions previously obtained by Mo and Tsai. Results are presented for some kinematics at high momentum transfer.

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I. INTRODUCTION

Electron scattering at intermediate and high energies has been one of the most useful means of investigating nuclear structure for over forty years. With the advent of CW accelerators and high resolution detectors such as MAMI and TJNAF it has become clear that one must have an accurate estimate of the radiative corrections if meaningful cross sections are to be obtained from the experimental measurements. Depending on the experimental conditions – initial beam energy, momentum transfer, and detector resolution or missing mass for the observed particles – the radiative corrections can be as large as 30% of the uncorrected cross section. To obtain cross sections which are accurate to 1%, one must then know the radiative correction to 3%.

The theoretical expression for the radiative correction which has been used in the analysis of almost all single arm elastic electron scattering experiments with beam energies below approximately 25 GeV (for which W and Z exchange are in general not significant) is that given originally by Tsai [1], [2] in connection with experiments at Stanford, SLAC and CEA. That work involved approximations that were both purely mathematical (made in performing the integrations needed to evaluate the inelastic cross section) and approximations denoted here as "soft-photon approximations" that are directly related to the physics in that the effect of proton structure was neglected; in considering the proton legs, only the soft virtual (infrared) photon contribution is calculated exactly - approximations are made in the hard virtual photon (non-infrared) contribution. In particular, the proton structure is neglected by setting the photon momentum square $k^2 = 0$ in the proton form factor, $F(k^2)$, thus simplifying the calculation considerably.

The purpose of the present paper is to study the radiative correction to elastic electron-proton scattering including the internal structure of the nucleon. For this we have considered a simple model in which the proton current is taken to have the usual on-shell form. The model dependence of the radiative correction is clearly an important question for the analysis of electron scattering experiments at the 1% level. This work is an initial study to examine the size of internal structure effects.

The present calculation differs from that of Tsai [1], [2] in three substantive aspects. First, we evaluate the inelastic cross section (emission of soft real photons) without any approximation; the relevant integrals have been given in closed form by t'Hooft and Veltman [3]. In fact, these exact expressions are simpler in form than the approximate ones given in [1] and [2]. We note in particular that in the limit of the target mass $M \to \infty$, corresponding to a static Coulomb potential, we obtain exactly the result first given by Schwinger [5]. Second, in the evaluation of the contribution of the box and crossed box diagrams to the elastic cross section we make a less drastic approximation than that made in [1]. Specifically, in the integrands corresponding to the relevant matrix elements, M_2 and M_3 (Eqs. (A5) and (A6)), we make a soft photon approximation (setting $k = 0$ or $k = q$) in the *numerator* (as in [1]), but not in the denominators. Again, the required integrals (scalar four-point functions) have been given in [3]; the resulting expressions are again considerably simpler than those obtained in [1], where the soft-photon approximation is also made in the denominators of M_2 and M_3. Finally, in evaluating the proton vertex correction, we have made no soft photon approximation for the virtual photon (as was done in [1]) and have included form factors for the proton, taking the proton current to be that indicated below in (2.4).

The organization of the paper is as follows: In Sec. II we discuss questions concerning the electromagnetic nuclear current operator used in this analysis. In Sec. III we give details of the calculation of the matrix elements and cross section for elastic scattering, retaining terms of order α relative to the Rosenbluth (one photon exchange) cross section for elastic scattering. Integrals needed for the evaluation of the various matrix elements are written explicitly and expressed in closed form in terms of Spence functions (dilogarithms). Details are given in the Appendices.

In Sec. IV we consider the inelastic cross section in detail; as with the elastic cross section given in Sec. III, the result is expressed in closed form in terms of Spence functions. In Sec. V we add the elastic and inelastic cross sections, giving both an analytic expression and a numerical evaluation of the radiative correction for various values of the pertinent parameters, (initial beam energy, final electron detector resolution, and target nucleus). We compare the values of the radiative correction calculated here with those given in [1] and [2].

II. ELECTROMAGNETIC NUCLEON CURRENT OPERATOR

We essentially follow in this paper the convention of Björken and Drell [8]. The metric used is defined by

$$p_i \cdot p_j = \epsilon_i \epsilon_j - \mathbf{p}_i \cdot \mathbf{p}_j \tag{2.1}$$

Further, $\alpha = e^2/4\pi = 1/137.036$; m is the electron rest mass; M is the target nucleus rest mass; Z the charge of the target nucleus; κ the anomalous magnetic moment of the proton; p_1 and p_3 the initial and final electron four-momenta, respectively; p_2 and p_4 the initial and final target nucleus four-momenta, respectively; $q = p_1 - p_3 = p_4 - p_2$ is the four-momentum transfer to the target nucleus for elastic scattering. It will prove useful to define, in addition,

$$\rho = p_4 + p_2, \;\; \rho_m = p_1 + p_3, \tag{2.2}$$

from which $\rho^2 = -q^2 + 4M^2$ and $\rho_m^2 = -q^2 + 4m^2$. Further, we define

$$\begin{aligned} x &= (\rho + \rho_1) \,/\, (\rho - \rho_1) = (\rho + \rho_1)^2 \,/\, 4M^2 \\ x_m &= (\rho_m + \rho_1) \,/\, (\rho_m - \rho_1) = (\rho_m + \rho_1)^2 \,/\, 4m^2 \end{aligned} \tag{2.3}$$

with $\rho_1^2 = -q^2$. In the lab system we have: $p_1 = (\epsilon_1, \mathbf{p}_1)$; $p_3 = (\epsilon_3, \mathbf{p}_3)$; $p_2 = (M, 0)$; $p_4 = (M + \omega, \mathbf{q})$; $\omega = -q^2/2M$.

With the aim of presenting expressions which correspond to the experimental conditions of high energy electron scattering, we neglect, in the final expressions given in this paper, terms of relative orders m^2/ϵ^2, $m^2/(-q^2)$, and m^2/M^2. Neglect of these terms defines our high energy approximation. No assumption is made, however, with regard to the magnitudes of M/ϵ_1, M/ϵ_3, or $M^2/(-q^2)$.

At low momentum transfer the internal structure of the nucleon can safely be neglected in the determination of the radiative corrections in electron-nucleus scattering. However, with increasing energies and momenta this is in general no longer the case. One of the objectives of this paper is to investigate this in a model for the e.m. interaction of a non-pointlike nucleon. The most general e.m. off-shell nucleon vertex can be characterized by 6 invariant functions [6,7]. As the most simple model we may consider a vector dominance-like model for the nucleon current, characterized by only two form factors which depend only on the four-momentum square of the photon. It is given by

$$\Gamma_\mu = F_1(q^2)\gamma_\mu + \kappa F_2(q^2)\frac{i\sigma_{\mu\nu}q^\nu}{2M}, \tag{2.4}$$

where the form factors $F_1(q^2)$ and $F_2(q^2)$ are taken to have a monopole or dipole form:

$$F_1(q^2) = F_2(q^2) = \left(\frac{-\Lambda^2}{q^2 - \Lambda^2}\right)^n, \qquad n = 1 \text{ or } 2 \tag{2.5}$$

with Λ being a constant of the order of 1 GeV/c. Furthermore, $q = p' - p$, p and p' being the momentum of the initial and final nucleon. Although the quantitative predictions of the radiative corrections are expected in general to be dependent on the details of the nucleon model assumed, one should already be able to see most of the salient features in the present model study. In particular, identifying regions in phase space where the finite size of the nucleon may play an important role in the size of radiative corrections can be important. In this way one may hope to get some feeling on the reliability of neglecting the internal structure of the nucleon as is usually done. The present study is intended as a first exploration of the sensitivity on the non-pointlike nature of the e.m. hadronic current. As in [1], although we are primarily interested in electron-proton scattering, the radiative corrections studied here can also be applied to electron-nucleus scattering, with appropriate changes in F_1, F_2, κ, and M. However, even in the case of electron-proton scattering, the factor Z is convenient for identifying the contributions from the various diagrams.

It should be noted that the dressed vertex function, $\widetilde{\Lambda}_\mu$, with Eq. (2.4) as e.m. current operator containing the form factors F_n, satisfies a Ward-Takahashi identity

$$q^\mu \widetilde{\Lambda}_\mu = F_1(q^2)\left[S^{-1}(p') - S^{-1}(p)\right] \tag{2.6}$$

where S is the dressed nucleon propagator. As a direct consequence of (2.6), one gets for on-mass-shell nucleons, the current conservation

$$q^\mu < p'|\widetilde{\Lambda}_\mu|p >= 0. \tag{2.7}$$

Obviously, the radiative corrections will in general be sensitive to the choice of the e.m. current. Although interesting in its own right we will not address in this paper the issue of the dependence of the predictions on this ambiguity.

In the study of radiative corrections we may distinguish between the elastic and inelastic contributions, the latter being the real soft photon emission processes from both the electron and hadron. The elastic electron cross section can be determined immediately from the total scattering amplitude $\mathcal{M}$ through the well-known expression

$$d\sigma = \frac{mM}{\sqrt{(p_1\cdot p_2)^2 - m^2M^2}} \sum_{spins} \int |\mathcal{M}|^2 \, (2\pi)^4 \delta^4(p_4 + p_3 - p_2 - p_1) \times \frac{m\, d^3p_3}{(2\pi)^3\epsilon_3}\, \frac{M\, d^3p_4}{(2\pi)^3\epsilon_4}, \tag{2.8}$$

For single-arm experiments with unpolarized electrons in which the final proton is not observed, $d\sigma$ must be averaged over initial spins, summed over final spins, and integrated over the final proton four-momentum. Up to order α^2 we have for the total scattering amplitude

$$\mathcal{M} = \sum_{n=1}^{6} M_n, \tag{2.9}$$

where the various terms correspond to the Feynman graph contributions shown in Fig. 1. M_1 is the matrix element for the one-photon exchange diagram

$$M_1 = Ze^2 \overline{u}(p_3)\gamma_\mu u(p_1) \frac{(-i)}{q^2 + i\epsilon} \overline{u}(p_4)\Gamma^\mu(q^2) u(p_2). \tag{2.10}$$

Its square gives, for high energy electrons, the Rosenbluth cross section:

$$\frac{d\sigma_0}{d\Omega} = \frac{\alpha^2 \cos^2\frac{\theta}{2}\left[\left(F_1^2 - \frac{\kappa^2 q^2}{4M^2}F_2^2\right) - \frac{q^2}{2M^2}\left(F_1 + \kappa F_2\right)^2 \tan^2\frac{\theta}{2}\right]}{4\epsilon_1^2 \eta \sin^4(\theta/2)}, \tag{2.11}$$

where η is the lab system recoil factor: For $\epsilon_1 >> m, \epsilon_3 >> m, \quad \eta \cong \epsilon_1/\epsilon_3 \cong 1+(\epsilon_1/M)(1-\cos\theta)$ with θ being the electron scattering angle. We note, in particular, that $1 \leq \eta \leq x$. Furthermore, M_2 and M_3 are the matrix elements for the box and crossed box (two-photon exchange) diagrams. M_4 is the vacuum polarization diagram (only an electron-positron loop is indicated in the figure, but the contribution from higher mass lepton loops can be included without difficulty- see (A8) -(A12)). M_5 is the electron vertex correction, and M_6 is the proton vertex correction. For completeness, we list in Appendix A the explicit expressions for the various Feynman diagrams shown in Figs. 1 and 2.

III. ELASTIC CROSS SECTION

To evaluate the various one-loop corrections to Eq. (2.9) some tedious algebra has to be carried out. We outline the procedure used to evaluate the matrix elements needed for the radiative correction to the elastic cross section, M_2 through M_6, (Eqs. (A1)-(A10)).

A. Proton vertex correction

We begin with the matrix element for the proton vertex correction, M_6 , given by Eqs. (A2) and (A4); the much simpler matrix element for the electron vertex correction, M_5, (Eq. (A1)), can be deduced quite easily from that. In (A4), each of the three Γ's, given by Eq. (2.4), contains a term with γ_μ (which we denote by g) and a term with $\sigma_{\mu\nu}$ (which we denote by s). The proton vertex correction $\Lambda^\mu(p_4, p_2)$ then consists of eight terms, which we represent symbolically by ggg, gsg, gss, etc. As may be seen after rationalizing the propagators, the k dependence of the numerators for $ggg, gsg, ...$ is such that there are at most four factors of the form $\not k$. Moreover, the terms with three or four factors $\not k$ may, with only a minimum of algebra, be written so that two of these factors are adjacent, giving $\not k \not k = k^2$. Although the calculation can equally well be carried out with F_1 and F_2 distinct functions, we assume $F_1 = F_2 = F$, which simplifies the algebra. The terms $ggg, gsg, ...$ can then be expressed in terms of the integrals

$$\{I_0; I_\mu; I_{\mu\nu}; J_0; J_\mu; J_{\mu\nu}; K_0\} = \int \frac{d^4k}{(2\pi)^4} F^2(k^2)\{1; k_\mu; k_\mu k_\nu; k^2; k_\mu k^2; k_\mu k_\nu k^2; (k^2)^2\}/D(\lambda^2) \tag{3.1}$$

where

$$D(\lambda^2) = (k^2 - \lambda^2 + i\epsilon)(k^2 - 2k\cdot p_2 + i\epsilon)(k^2 - 2k\cdot p_4 + i\epsilon) \tag{3.2}$$

For form factors having the form given in (2.5), the integrals in (3.1) could all be evaluated as indicated for three-point functions in [3], Sec.5, and [4], Appendix E. However, in the interest of obtaining a relatively compact analytic expression in closed form, we have used an alternative procedure. As given here in Appendix B, the integrals may be expressed in terms of their moments, defined by Eqs. (B4)-(B6) and (B10). After straightforward though somewhat tedious algebra, the terms $ggg, gsg, ...$ are then expressed in terms of these moments. Next, for form factors of the form given in (2.5), we show that all of the moments may be expressed in terms of three functions, ϕ_k, which obey a three-term inhomogenous recursion, and this is used for their evaluation. Finally, we note from (B51) - (B57) that the terms $ggg, gsg, ...$ may be usefully grouped by writing them in the form

$$(g+s)g(g+s) = F(q^2)\left[G_1(q^2)\gamma_\mu + G_2(q^2)\frac{i\sigma_{\mu\nu}q^\nu}{2M}\right] \tag{3.3}$$

and

$$(g+s)s(g+s) = \kappa F(q^2)\left[X_1(q^2)\gamma_\mu + X_2(q^2)\frac{i\sigma_{\mu\nu}q^\nu}{2M}\right] \tag{3.4}$$

We note in the expressions for ggg, gsg,... in Appendix B that the infrared divergent terms are all contained solely within ggg and gsg. These are the terms with a factor $\phi_1(\lambda^2)$ in (B51) and (B52). Since these are precisely the terms which are retained in the proton vertex correction in [1], we separate them for the purpose of comparison with that work, writing M_6 in the form

$$M_6 = M_6^{(0)} + M_6^{(1)} \tag{3.5}$$

where

$$M_6^{(0)} = -\frac{\alpha Z^2}{2\pi}(2M^2 - q^2)\phi_1(\lambda^2)M_1 \tag{3.6}$$

The function $\phi_1(\lambda^2)$, defined by (B17), (B25) and (B26) and evaluated in (B33) and (B37), is simply related to the function $K(p_2, p_4)$ defined in [1] by

$$K(p_i, p_j) = \frac{2p_i \cdot p_j}{-i\pi^2} \int \frac{d^4k}{(k^2 - \lambda^2 + i\epsilon)(k^2 - 2k \cdot p_i + i\epsilon)(k^2 - 2k \cdot p_j + i\epsilon)}$$

viz.,

$$K(p_2, p_4) = 2p_2 \cdot p_4 \phi_1(\lambda^2) \tag{3.7}$$

B. Proton self energy correction

We next consider the contribution of the proton self energy diagrams. It is given by Σ' where

$$\Sigma' = \frac{1}{4} Tr \left[\frac{\partial \Sigma}{\partial \not{p}}\right] \tag{3.8}$$

in which Σ is the lowest order self-energy contribution

$$\Sigma = -ie^2 \int \frac{d^4k}{(2\pi)^4} \frac{1}{k^2 - \lambda^2 + i\epsilon} \Gamma^\nu(k) \frac{1}{(\not{p} - \not{k} - M + i\epsilon)} \Gamma_\nu(k) \tag{3.9}$$

Using the Ward-Takahashi identity Eq. (2.6) we find that

$$\Sigma' = G_1(0)$$

where $G_1(q^2)$ is the coefficient of γ_μ in ggg (B51). Explicitly, $G_1(0)$ is given by Eqs.(B29) - (B32) and (B38). The addition of this contribution to the lowest order vertex correction modifies the expressions for $(g+s)g(g+s)$ and $(g+s)s(g+s)$ given above in (3.3) and (3.4) so that we now have

$$\overline{(g+s)g(g+s)} = F(q^2)\left[\left(G_1(q^2) - G_1(0)\right)\gamma_\mu + G_2(q^2)\frac{i\sigma_{\mu\nu}q^\nu}{2M}\right] \tag{3.10}$$

and

$$\overline{(g+s)s(g+s)} = \kappa F(q^2)\left[X_1(q^2)\gamma_\mu + \left(X_2(q^2) - G_1(0)\right)\frac{i\sigma_{\mu\nu}q^\nu}{2M}\right] \tag{3.11}$$

We then have, for the matrix element including self energy diagrams,

$$\overline{M}_6 = \overline{M}_6^{(0)} + \overline{M}_6^{(1)} \tag{3.12}$$

where

$$\overline{M}_6^{(0)} = \frac{\alpha Z^2}{2\pi}\left[-K(p_2,p_4) + K(p_2,p_2)\right] M_1 \tag{3.13}$$

which is the expression given in [1], eq. (II.12). The infrared divergent part of these terms is cancelled exactly by the infrared divergent terms in the inelastic cross section. The contribution of the matrix element $M_6^{(1)}$, which depends on the proton form factor, will be considered after we write the electron vertex and box diagram corrections.

C. Electron vertex correction

The electron vertex correction, M_5, may be obtained directly from the proton vertex correction, M_6. The expression $\Lambda_\mu(p_3,p_1)$, Eq. (A3), follows from $\Lambda^\mu(p_4,p_2)$ if we retain only the term ggg, set $F=1$, replace p_2, p_4 and M by p_1, p_3 and m, and, after performing the integrations, take the limit $\Lambda \to \infty$ (note Eq.(2.5)). Note that $\rho = p_4 + p_2$ is then replaced by $\rho_m = p_3 + p_1$, and x, defined in section II, is replaced by x_m. Details are given in Appendix C. We find

$$ggg = G_1^{(e)}(q^2)\gamma_\mu + G_2^{(e)}(q^2)\frac{i\sigma_{\mu\nu}q^\nu}{2m} \tag{3.14}$$

where

$$G_1^{(e)}(q^2) = \frac{\alpha}{2\pi}\left\{-K(p_1,p_3) + \left(\frac{3\rho_1^2 + 8m^2}{2\rho_m\rho_1}\right)\ln x_m + \frac{1}{4} + \frac{1}{2}\ln\left(\frac{\Lambda^2}{m^2}\right)\right\} \tag{3.15}$$

$$G_2^{(e)}(q^2) = \frac{\alpha}{2\pi}\left\{\frac{2m^2}{\rho_m\rho_1}\ln x_m\right\} \tag{3.16}$$

Adding the contribution of the electron self energy diagrams gives

$$\overline{ggg} = \left(G_1^{(e)}(q^2) - G_1^{(e)}(0)\right)\gamma_\mu + G_2^{(e)}(q^2)\frac{i\sigma_{\mu\nu}q^\nu}{2m} \tag{3.17}$$

from which

$$\overline{ggg} = \frac{\alpha}{2\pi}\left\{\left(-K(p_1,p_3) + K(p_1,p_1) + \frac{3\rho_1^2 + 8m^2}{2\rho_m\rho_1}\ln x_m - 2\right)\gamma_\mu + \left(\frac{2m^2}{\rho_m\rho_1}\ln x\right)\frac{i\sigma_{\mu\nu}q^\nu}{2m}\right\} \tag{3.18}$$

For large momentum transfers, $-q^2 \gg m^2$, this reduces to

$$\overline{ggg} = \frac{\alpha}{2\pi}\left\{-K(p_1,p_3)+K(p_1,p_1)+\frac{3}{2}\ln\left(\frac{-q^2}{m^2}\right)-2\right\}\gamma_\mu \tag{3.19}$$

Comparing (2.10) and (A1), we then have, for $-q^2 \gg m^2$,

$$\overline{M}_5 = \frac{\alpha}{2\pi}\left\{-K(p_1,p_3)+K(p_1,p_1)+\frac{3}{2}\ln\left(\frac{-q^2}{m^2}\right)-2\right\}M_1 \tag{3.20}$$

which is the expression given in [1], eq.(II.5). We note that the infrared divergence is contained entirely within the terms $-K(p_1,p_3)+K(p_1,p_1)$. The infrared divergent part of these terms is cancelled exactly by the infrared divergent terms in the inelastic cross section.

D. Box and crossed-box diagrams

The matrix elements for the box and crossed-box diagrams, M_2 and M_3, are given in (A5) and (A6). After rationalizing the propagators, the required integrals can, for form factors of the form (2.5), all be written in terms of four-point functions; in principle they can be evaluated using [3], Sec. 6, and [4], Appendix E. For the present work, however, we have chosen to evaluate these matrix elements in an approximate manner, but one which is less drastic than that employed in [1]. We note first in M_2 and M_3 that the integrands in M_2 and M_3 have two infrared divergent factors, $[(k^2-\lambda^2+i\epsilon)((k-q)^2-\lambda^2+i\epsilon)]^{-1}$. The integrands are thus peaked when either of the two exchanged photons is soft, and become divergent when $k \to 0$ or when $k \to q$. We therefore evaluate the numerators in M_2 and M_3 at these two points but make no changes to the denominators. A simple calculation shows that in fact that each of the numerators has the same value for $k=0$ as for $k=q$, *viz.*, $4ip_1\cdot p_2\, q^2 M_1$ in the case of M_2 and $4ip_3\cdot p_2\, q^2 M_1$ in the case of M_3. We then take this factor outside of the integral and are left with a scalar four-point function to evaluate. The result has been given in [3], Sec. 6 and Appendix E (b) and is expressed simply in terms of logarithms:

$$M_2 = -\frac{\alpha Z}{\pi}\frac{\epsilon_1}{|\mathbf{p_1}|}\ln\left(\frac{\epsilon_1+|\mathbf{p_1}|}{m}\right)\ln\left(\frac{-q^2}{\lambda^2}\right)M_1 \tag{3.21}$$

and

$$M_3 = \frac{\alpha Z}{\pi}\frac{\epsilon_3}{|\mathbf{p_3}|}\ln\left(\frac{\epsilon_3+|\mathbf{p_3}|}{m}\right)\ln\left(\frac{-q^2}{\lambda^2}\right)M_1 \tag{3.22}$$

By contrast, in [1], in addition to the approximation just described, a soft-photon approximation is made in the infrared denominators: Specifically, when $k=0$ the factor $(k-q)^2-\lambda^2$ is set equal to $q^2-\lambda^2$ and when $k=q$ the factor $k^2-\lambda^2$ is set equal to $q^2-\lambda^2$, thus giving two terms and reducing the four-point function to three-point functions:

$$M_2 = -\frac{\alpha Z}{2\pi}\left[K(p_2,-p_1)+K(p_4,-p_3)\right]M_1 \tag{3.23}$$

and

$$M_3 = \frac{\alpha Z}{2\pi}\left[K(p_2,p_3) + K(p_4,p_1)\right] M_1 \tag{3.24}$$

(Note [1], eqs.(II.9) and (II.11)). The infrared divergent terms (those with a factor $\ln \lambda^2$) are, for M_2, the same in (3.21) and (3.23), and, for M_3, the same in (3.22) and (3.24) . However, (3.23) and (3.24) differ significantly from (3.21) and (3.22). These latter expressions are functions of the momentum transfer, q^2. The integrals $K(p_i, p_j)$, on the other hand, are functions only of the scalar invariants p_i^2, p_j^2 and $p_i \cdot p_j$. In (3.21) and (3.22), M_2 and M_3 therefore depend only on the initial and final electron energies, and not on the momentum transfer ($p_2 \cdot p_1 = p_4 \cdot p_3 = \epsilon_1 M$; $p_2 \cdot p_3 = p_4 \cdot p_1 = \epsilon_3 M$).

From (A8)-(A10), (3.12), (3.13), (3.20), (3.21) and (3.22), we can now write the square of the matrix element for elastic scattering, including the radiative correction to order α. Assuming $-q^2 \gg m^2$, and including only electron-positron pairs in the vacuum polarization matrix element, we have

$$|\mathcal{M}|^2 = |M_1|^2 \left\{ \begin{array}{c} 1 + \frac{\alpha}{\pi}\left[\frac{13}{6}\ln\left(\frac{-q^2}{m^2}\right) - \frac{28}{9} - K(p_1,p_3) + K(p_1,p_1)\right] \\ -\frac{2\alpha Z}{\pi}\ln\eta\ln\left(\frac{-q^2}{\lambda^2}\right) \\ +\frac{\alpha Z^2}{\pi}\left[-K(p_2,p_4) + K(p_2,p_2)\right] \end{array} \right\} \tag{3.25}$$
$$+2\mathrm{Re}\left\{M_1^\dagger \overline{M}_6^{(1)}\right\}$$

E. Contribution of proton form factor

Finally, we consider the contribution of the term $2\mathrm{Re}\left\{M_1^\dagger \overline{M}_6^{(1)}\right\}$, coming from the inclusion of form factors for the proton and integration over the entire range of four-momenta of the virtual photon in the proton vertex correction. Equations (3.3) and (3.4) define the functions $G_1(q^2), G_2(q^2), X_1(q^2)$, and $X_2(q^2)$. From (3.5) we may write $M_6^{(1)} = M_6 - M_6^{(0)}$, *i.e.*, the term $M_6^{(1)}$ is obtained from the full proton vertex correction by subtracting the infrared divergent matrix element $M_6^{(0)}$ which is independent of the proton form factor. We therefore define $G_1^{'}(q^2)$ and $X_2^{'}(q^2)$ to be the expressions $G_1(q^2)$ and $X_2(q^2)$ from which we have omitted the terms with factor $\phi_1(\lambda^2)$ We then write, from (3.3) and (3.4),

$$\overline{(g+s)g(g+s)}\,' = F(q^2)\left[\left(G_1^{'}(q^2) - G_1^{'}(0)\right)\gamma_\mu + G_2(q^2)\frac{i\sigma_{\mu\nu}q^\nu}{2M}\right] \tag{3.26}$$

and

$$\overline{(g+s)s(g+s)}\,' = \kappa F(q^2)\left[X_1(q^2)\gamma_\mu + \left(X_2^{'}(q^2) - G_1^{'}(0)\right)\frac{i\sigma_{\mu\nu}q^\nu}{2M}\right] \tag{3.27}$$

We now define the functions $F_{1g}(q^2), F_{2g}(q^2), F_{1s}(q^2)$, and $F_{2s}(q^2)$ by

$$F(q^2)\left[\left(G_1^{'}(q^2) - G_1^{'}(0)\right)\gamma_\mu + G_2(q^2)\frac{i\sigma_{\mu\nu}q^\nu}{2M}\right] \equiv F_{1g}(q^2)\gamma_\mu + \kappa F_{2g}(q^2)\frac{i\sigma_{\mu\nu}q^\nu}{2M} \tag{3.28}$$

$$\kappa F(q^2)\left[X_1(q^2)\gamma_\mu + \left(X_2'(q^2\} - G_1'(0)\right)\frac{i\sigma_{\mu\nu}q^\nu}{2M}\right] \equiv F_{1s}(q^2)\gamma_\mu + \kappa F_{2s}(q^2)\frac{i\sigma_{\mu\nu}q^\nu}{2M} \tag{3.29}$$

Then with the further definitions

$$\widetilde{F}_1(q^2) \equiv F_{1g}(q^2) + F_{1s}(q^2) \tag{3.30}$$

$$\widetilde{F}_2(q^2) \equiv F_{2g}(q^2) + F_{2s}(q^2) \tag{3.31}$$

$$\widetilde{\Gamma}_\mu \equiv \widetilde{F}_1(q^2)\gamma_\mu + \kappa\widetilde{F}_2(q^2)\frac{i\sigma_{\mu\nu}q^\nu}{2M} \tag{3.32}$$

we have (apart from factors)

$$\overline{M}_6^{(1)} = \frac{\alpha Z^2}{2\pi}\langle p_3|\gamma^\mu|p_1\rangle\left\langle p_4|\widetilde{\Gamma}_\mu|p_2\right\rangle \tag{3.33}$$

and

$$2\mathrm{Re}\left\{M_1^\dagger\overline{M}_6^{(1)}\right\} = \frac{\alpha Z^2}{\pi}\left(\langle p_3|\gamma^\nu|p_1\rangle\langle p_4|\Gamma_\nu|p_2\rangle\right)^\dagger\left(\langle p_3|\gamma^\mu|p_1\rangle\left\langle p_4|\widetilde{\Gamma}_\mu|p_2\right\rangle\right) \tag{3.34}$$

This has the same form as

$$M_1^\dagger M_1 = \left(\langle p_3|\gamma^\nu|p_1\rangle\langle p_4|\Gamma_\nu|p_2\rangle\right)^\dagger\left(\langle p_3|\gamma^\mu|p_1\rangle\langle p_4|\Gamma_\mu|p_2\rangle\right) \tag{3.35}$$

with the exception of the replacement $\Gamma_\mu \to \widetilde{\Gamma}_\mu$ in the right-hand term. Thus, in place of the Rosenbluth cross section, (2.11), obtained from $\sum_{spins} M_1^\dagger M_1$, we have

$$\sum_{spins} 2\mathrm{Re}\left\{M_1^\dagger\overline{M}_6^{(1)}\right\} = \frac{\alpha^2\cos^2\frac{\theta}{2}}{4\epsilon_1^2\eta\sin^4(\theta/2)}\left(\frac{\alpha Z^2}{\pi}\right)\left\{\ \right\} \tag{3.36}$$

where

$$\left\{\ \right\} = \left(F_1\widetilde{F}_1 - \frac{\kappa^2 q^2}{4M^2}F_2\widetilde{F}_2\right) - \frac{q^2}{2M^2}\left(F_1+\kappa F_2\right)\left(\widetilde{F}_1+\kappa\widetilde{F}_2\right)\tan^2\frac{\theta}{2} \tag{3.37}$$

The purely elastic cross section, including radiative corrections to order α, can thus be written as

$$\left(\frac{d\sigma_0}{d\Omega}\right)\left(1+\delta_{el}^{(0)}+\delta_{el}^{(1)}\right) \tag{3.38}$$

where

$$\begin{aligned}\delta_{el}^{(0)} &= \frac{\alpha}{\pi}\left[-\left[\ln\left(\frac{-q^2}{m^2}\right)-1\right]\ln\left(\frac{m^2}{\lambda^2}\right)+\frac{13}{6}\ln\left(\frac{-q^2}{m^2}\right)-\frac{28}{9}-\frac{1}{2}\ln^2\left(\frac{-q^2}{m^2}\right)+\frac{\pi^2}{6}\right]\\ &- \frac{2\alpha Z}{\pi}\ln\eta\ln\left(\frac{-q^2}{\lambda^2}\right)\\ &+ \frac{\alpha Z^2}{\pi}\left[-\left(\frac{\epsilon_4}{|\mathbf{p_4}|}\ln x - 1\right)\ln\left(\frac{M^2}{\lambda^2}\right)+\frac{\epsilon_4}{|\mathbf{p_4}|}\left[-\ln x\ln\left(\frac{\rho^2}{M^2}\right)+\frac{1}{2}\ln^2 x + 2L(-\frac{1}{x})+\frac{\pi^2}{6}\right]\right]\end{aligned} \tag{3.39}$$

and

$$\delta_{el}^{(1)} = \frac{\alpha Z^2}{\pi}\left\{\frac{\left(F_1\widetilde{F}_1 - \frac{\kappa^2q^2}{4M^2}F_2\widetilde{F}_2\right) - \frac{q^2}{2M^2}\left(F_1+\kappa F_2\right)\left(\widetilde{F}_1+\kappa\widetilde{F}_2\right)\tan^2\frac{\theta}{2}}{\left[\left(F_1^2-\frac{\kappa^2q^2}{4M^2}F_2^2\right)-\frac{q^2}{2M^2}\left(F_1+\kappa F_2\right)^2\tan^2\frac{\theta}{2}\right]}\right\} \tag{3.40}$$

IV. INELASTIC CROSS SECTION

In this section we calculate the inelastic cross section, i.e., the contribution of soft photon emission by the initial and final electron and proton to the radiative correction. The relevant diagrams, with corresponding matrix elements M_{b1} and M_{b2}, are shown in Fig. 2. These matrix elements are given by

$$\begin{aligned} M_{b1} = & -iZe^3(2\pi)^4\delta^4(p_3+p_4+k-p_1-p_2)\frac{mM}{\sqrt{2\omega\epsilon_1\epsilon_3\epsilon_2\epsilon_4}} \\ & \times\overline{u}(p_3)\left[\not\epsilon\frac{1}{\not p_3+\not k-m+i\epsilon}\gamma_\mu+\gamma_\mu\frac{1}{\not p_1-\not k-m+i\epsilon}\not\epsilon\right]u(p_1) \\ & \times\overline{u}(p_4)\Gamma^\mu u(p_2)\frac{1}{(p_1-p_3-k)^2+i\epsilon} \end{aligned} \tag{4.1}$$

$$\begin{aligned} M_{b2} = & \, iZ^2e^3(2\pi)^4\delta^4(p_3+p_4+k-p_1-p_2)\frac{mM}{\sqrt{2\omega\epsilon_1\epsilon_3\epsilon_2\epsilon_4}}\overline{u}(p_3)\gamma_\mu u(p_1) \\ & \times\overline{u}(p_4)\left[\not\epsilon\frac{1}{\not p_4+\not k-m+i\epsilon}\Gamma^\mu+\Gamma^\mu\frac{1}{\not p_2-\not k-m+i\epsilon}\not\epsilon\right]u(p_2)\frac{1}{(p_1-p_3)^2+i\epsilon} \end{aligned} \tag{4.2}$$

Making the soft photon approximation, we rationalize the denominators and drop terms of relative order k in the numerator and denominator (but not in the delta function), giving

$$\begin{aligned} M_{b1}+M_{b2} = & -iZe^3(2\pi)^4\delta^4(p_3+p_4+k-p_1-p_2)\frac{mM}{\sqrt{2\omega\epsilon_1\epsilon_3\epsilon_2\epsilon_4}}\frac{1}{q^2} \\ & \times\overline{u}(p_3)\gamma_\mu u(p_1)\,\overline{u}(p_4)\Gamma^\mu u(p_2)\left(\frac{p_3\cdot\epsilon}{p_3\cdot k}-\frac{p_1\cdot\epsilon}{p_1\cdot k}-Z\frac{p_4\cdot\epsilon}{p_4\cdot k}+Z\frac{p_2\cdot\epsilon}{p_2\cdot k}\right) \end{aligned} \tag{4.3}$$

The cross section for soft bremsstrahlung then follows by squaring the matrix element $M_{b1}+M_{b2}$, dividing by the incident flux and the transition rate and multiplying by the number of final states. Summing over photon polarizations, we then have

$$\begin{aligned} d\sigma_b = & -\frac{Z^2e^6}{(2\pi)^9}\frac{m^2M^2}{\sqrt{(p_1\cdot p_2)^2-m^2M^2}}\sum_{spins}\int\frac{d^3p_3}{\epsilon_3}\frac{d^3p_4}{\epsilon_4}\frac{d^3k}{2\omega}(2\pi)^4\delta^4(p_3+p_4+k-p_1-p_2) \\ & \times\left|\overline{u}(p_3)\gamma_\mu u(p_1)\,\overline{u}(p_4)\Gamma^\mu u(p_2)\right|^2\frac{1}{(q^2)^2}\left(\frac{p_3}{p_3\cdot k}-\frac{p_1}{p_1\cdot k}-Z\frac{p_4}{p_4\cdot k}+Z\frac{p_2}{p_2\cdot k}\right)^2 \end{aligned} \tag{4.4}$$

The range of integration in the above expression is determined by the experimental conditions. We assume, as in [1], that the final proton and emitted photon are undetected; the range of integration in energy and angle of the final electron is determined by the entrance slit and spectrometer. We integrate first over d^3p_4 and are then left with a single delta function relating the variables of $\mathbf{k}$ and $\mathbf{p}_3$: Writing

$$\frac{d^3p_4}{2\epsilon_4}=\int_0^\infty d\epsilon_4\,\delta(p_4^2-M^2)\,d^3p_4=\int d^4p_4\,\delta(p_4^2-M^2)\,\theta(p_4^0) \tag{4.5}$$

with

$$t \equiv p_1 + p_2 - p_3 = p_4 + k \tag{4.6}$$

we then have

$$d\sigma_b = -\frac{Z^2 e^6}{(2\pi)^5} \frac{m^2 M^2}{\sqrt{(p_1 \cdot p_2)^2 - m^2 M^2}} \sum_{spins} \int \frac{d^3 p_3}{\epsilon_3} \int \frac{d^3 k}{\omega} \delta\left((t-k)^2 - M^2\right) \theta(\epsilon_4)$$
$$\times \left|\overline{u}(p_3)\gamma_\mu u(p_1)\, \overline{u}(p_4)\Gamma^\mu u(p_2)\right|^2 \frac{1}{(q^2)^2} \left(\frac{p_3}{p_3 \cdot k} - \frac{p_1}{p_1 \cdot k} - Z\frac{p_4}{p_4 \cdot k} + Z\frac{p_2}{p_2 \cdot k}\right)^2 \tag{4.7}$$

in which $p_4 = p_1 + p_2 - p_3 - k$. We may then transform to the special frame S^0 (defined by $\mathbf{t} = 0$), in which the delta function in (4.7) is independent of the angle at which the photon is emitted. There

$$(t-k)^2 - M^2 = t_0^2 - 2t_0\omega + \lambda^2 - M^2 = 0; \qquad\qquad t_0 = \epsilon_1 + \epsilon_2 - \epsilon_3 \tag{4.8}$$

The photon energy is then given solely by the final electron energy. The procedure used in [1] is to integrate next over the photon energy and angle in S^0 and then transform back to the lab frame to integrate over the energy and angle of the final electron. Instead, we remain in the special frame and integrate first over ϵ_3, the delta function giving ϵ_3 in terms of ω. However, the range of photon energies is assumed to be sufficiently small compared to all other energies that we can set ϵ_3 equal to its value for elastic scattering throughout the integrand. In addition, we take the angular range of the final electron to be sufficiently small that we can take some average value for these angles and neglect any variation of the integrand over this angular range. Similarly, we neglect k in the above expression for p_4. We may then take $\left|\overline{u}(p_3)\gamma_\mu u(p_1)\, \overline{u}(p_4)\Gamma^\mu u(p_2)\right|^2 \big/ (q^2)^2$ outside of the integration, giving

$$d\sigma_b = -\frac{Z^2 e^6}{(2\pi)^5} \frac{m^2 M^2}{\sqrt{(p_1 \cdot p_2)^2 - m^2 M^2}} \sum_{spins} \left|\overline{u}(p_3)\gamma_\mu u(p_1)\, \overline{u}(p_4)\Gamma^\mu u(p_2)\right|^2 \frac{1}{(q^2)^2}$$
$$\times \frac{|\mathbf{p}_3|}{2M} \int' \frac{d^3 k}{\omega} \left(\frac{p_3}{p_3 \cdot k} - \frac{p_1}{p_1 \cdot k} - Z\frac{p_4}{p_4 \cdot k} + Z\frac{p_2}{p_2 \cdot k}\right)^2 \tag{4.9}$$

The term $2M$ in the denominator in (4.9) comes from the delta function in (4.7), which contributes the factor $|d\{\delta\left((t-k)^2 - M^2\right)\}/d\epsilon_3| = 2|t_0 - \omega|$ from (4.8). Again neglecting terms of order k, we note from (4.6) that in S^0, $t_0 - \omega = \epsilon_4 = M$. Comparing (4.4), (2.8) and (2.10) and noting that in arriving at (4.9) we have neglected terms of order k in p_3 and p_4, we have

$$d\sigma_b = -\frac{\alpha}{4\pi^2} d\sigma_0 \int' \frac{d^3 k}{\omega} \left(\frac{p_3}{p_3 \cdot k} - \frac{p_1}{p_1 \cdot k} - Z\frac{p_4}{p_4 \cdot k} + Z\frac{p_2}{p_2 \cdot k}\right)^2 \tag{4.10}$$

where $\omega = \sqrt{\mathbf{k}^2 + \lambda^2}$. There then remains the integration over photon energy (restricted to $|\mathbf{k}| \leq \Delta\epsilon$) and angles. The relevant integrals have been evaluated by 't Hooft and Veltman [3], Sec. 7. We give here only their final result, rewritten using our metric; the essential steps in the derivation are given in their work. They define

$$L_{ij} = \int' \frac{d^3k}{\omega} \frac{1}{(p_i \cdot k)(p_j \cdot k)} \tag{4.11}$$

in terms of which

$$d\sigma_b = -\frac{\alpha}{4\pi^2} d\sigma_0 \left\{ \begin{array}{c} m^2 L_{11} + m^2 L_{33} - 2p_1 \cdot p_3 L_{13} \\ +Z\left(-2p_1 \cdot p_2 L_{12} + 2p_3 \cdot p_2 L_{32} + 2p_1 \cdot p_4 L_{14} - 2p_3 \cdot p_4 L_{34}\right) \\ +Z^2\left(M^2 L_{22} + M^2 L_{44} - 2p_2 \cdot p_4 L_{24}\right) \end{array} \right\} \tag{4.12}$$

As shown in [3], Sec.7, for the case in which the momenta p_i and p_j are all on the mass shell, the integrals L_{ij} can, provided p_i is not a multiple p_j, be written in the form

$$L_{ij} = \frac{2\pi}{\sqrt{(p_i \cdot p_j)^2 - m_i^2 m_j^2}} \left\{ S_{ij}^{(1)} + S_{ij}^{(2)} \right\} \tag{4.13}$$

where

$$S_{ij}^{(1)} = 2 \ln \left(\frac{p_i \cdot p_j + \sqrt{(p_i \cdot p_j)^2 - m_i^2 m_j^2}}{m_i m_j} \right) \ln \left(\frac{2\Delta\epsilon}{\lambda} \right) \tag{4.14}$$

and

$$\begin{aligned} S_{ij}^{(2)} = & \ln^2 \left(\frac{\beta_i}{m_i \sqrt{t^2}} \right) - \ln^2 \left(\frac{\beta_j}{m_j \sqrt{t^2}} \right) \\ & + L\left(1 - \frac{\beta_i\, l \cdot t}{t^2 \gamma_{ij}}\right) + L\left(1 - \frac{m_i^2\, l \cdot t}{\beta_i \gamma_{ij}}\right) \\ & - L\left(1 - \frac{\beta_j\, l \cdot t}{\alpha t^2 \gamma_{ij}}\right) - L\left(1 - \frac{m_j^2\, l \cdot t}{\alpha \beta_j \gamma_{ij}}\right) \end{aligned} \tag{4.15}$$

in which

$$\alpha = \frac{p_i \cdot p_j + \sqrt{(p_i \cdot p_j)^2 - m_i^2 m_j^2}}{m_i^2}; \qquad l = \alpha p_i - p_j \tag{4.16}$$

$$\beta_{i,j} \equiv p_{i,j} \cdot t + \sqrt{(p_{i,j} \cdot t)^2 - m_{i,j}^2 t^2}; \qquad \gamma_{ij} \equiv \sqrt{(p_i \cdot p_j)^2 - m_i^2 m_j^2} \tag{4.17}$$

The evaluation of (4.11) for $p_i = p_j$ is straightforward. The result written in terms of relativistic invariants is

$$L_{ii} = \frac{4\pi}{m_i^2} \left[\ln \left(\frac{2\Delta\epsilon}{\lambda} \right) - \frac{p_i \cdot t}{\sqrt{(p_i \cdot t)^2 - m_i^2 t^2}} \ln \left(\frac{\beta_i}{m_i \sqrt{t^2}} \right) \right] \tag{4.18}$$

In [3], Sec. 7, the expression for $S_{ij}^{(2)}$ is evaluated in the frame S^0, defined by $\mathbf{t} = 0$. Since we want finally to express the cross section in terms of lab frame energies and momenta, we have, in (4.15), written $S_{ij}^{(2)}$ in terms of relativistic invariants.

The terms of leading order in $\ln\lambda$ are apparent in (4.14) and (4.18). Substituting these in (4.12) gives the infrared divergent terms in $d\sigma_b$. They are cancelled exactly by the $\ln\lambda$ terms in the elastic cross section.

We next express $d\sigma_b$ in terms of lab frame energies. To that end, we assume that $\Delta\epsilon$ is less than any of the other energies and therefore now neglect k in (4.6), taking p_4 to be given by its value for elastic (non-radiative) scattering:

$$t = p_1 + p_2 - p_3 = p_4 \tag{4.19}$$

(Note that for $\Delta\epsilon \to 0$, $S^{(2)}_{ij}$ remains finite; the only singularity is confined to the term $\ln(2\Delta\epsilon/\lambda)$, evident in $S^{(1)}_{ij}$.) The relativistic invariants in L_{ij} can then be written simply in terms of lab frame energies:

$$\begin{aligned}
p_1\cdot p_4 &= p_2\cdot p_3 = M\epsilon_3\\
p_3\cdot p_4 &= p_2\cdot p_1 = M\epsilon_1\\
p_1\cdot p_3 &= -\frac{1}{2}q^2 + m^2\\
p_2\cdot p_4 &= -\frac{1}{2}q^2 + M^2
\end{aligned} \tag{4.20}$$

Further, we express $\Delta\epsilon$, the maximum momentum of the photon in the frame S^0, in terms of the final electron detector acceptance in the lab frame, ΔE:

$$\Delta\epsilon = \eta\Delta E \tag{4.21}$$

Details of the derivation are given in Appendix E. Substituting (4.19) and (4.20) in (4.14), (4.15) and (4.18) then gives L_{ij} in terms of lab frame energies and momenta. We note that although L_{ij} as defined in (4.11) is clearly symmetric in i and j, the expression for $S^{(2)}_{ij}$ in (4.15) is not manifestly symmetric; the form of the expression for L_{ij} is rather different from that for L_{ji}. In writing the explicit expressions for the terms in L_{ij}, we choose i and j such that L_{ij} simplifies readily for lab frame electron energies and momentum transfers which are very large compared to the electron rest mass. When $m_i \neq m_j$, this is achieved by choosing i and j such that $m_i = m$ and $m_j = M$ (note below that we have written $S^{(2)}_{32}$).

At this point we make the high energy approximation as defined in section II, in which case the above expressions for $S^{(1)}_{ij}$ and $S^{(2)}_{ij}$ simplify considerably. For $S^{(1)}_{ij}$ this is straightforward. We give the results for $S^{(2)}_{ij}$ in Appendix F. Substituting these in L_{ij}, the high energy approximation for the inelastic cross section given in (4.12) then becomes

$$d\sigma_b = \frac{\alpha}{\pi}\,d\sigma_0 \left\{ \begin{array}{c}
\left[\ln\left(\frac{-q^2}{m^2}\right) - 1\right]\ln\left(\frac{(2\eta\Delta E)^2}{\lambda^2}\right)\\
-\left[\ln\left(\frac{-q^2}{m^2}\right) - 1\right]\ln\left(\frac{4\epsilon_1\epsilon_3}{m^2}\right)\\
+\frac{1}{2}\ln^2\left(\frac{-q^2}{m^2}\right) - \frac{1}{2}\ln^2\eta + L(\cos^2\frac{1}{2}\theta) - \frac{1}{3}\pi^2\\
+2Z\left[\begin{array}{c} \ln\eta\,\ln\left(\frac{(2\eta\Delta E)^2}{\lambda^2}\right) - \ln\eta\,\ln x\\ +L(1-\frac{\eta}{x}) - L(1-\frac{1}{\eta x}) \end{array}\right]\\
+Z^2\left[\begin{array}{c} \left(\frac{\epsilon_4}{|\mathbf{p}_4|}\ln x - 1\right)\ln\left(\frac{(2\eta\Delta E)^2}{\lambda^2}\right)\\ -\frac{\epsilon_4}{|\mathbf{p}_4|}\left[\ln^2 x - \ln x + L(1-\frac{1}{x^2})\right] - 1 \end{array}\right]
\end{array}\right\} \tag{4.22}$$

V. RADIATIVE CORRECTIONS TO ELASTIC ELECTRON-PROTON SCATTERING

The results given in (3.39), (3.40), and (4.22) may be added to give the radiative correction, δ. The analytic expression is given below in (5.1) and (5.2). Numerical evaluation of the radiative correction for various values of the pertinent parameters (initial beam energy, momentum transfer, final electron detector resolution, and target nucleus) are given in Tables I, II, and III. We note that the infrared ($\ln\lambda$) terms, which appear in both the purely elastic (3.39) and inelastic (4.22) contributions to the radiative correction, cancel exactly when added to give the cross section for elastic electron-proton scattering with radiative corrections to first order in α:

$$d\sigma = d\sigma_0(1+\delta) \tag{5.1}$$

where

$$\begin{aligned}
\delta = {} & \frac{\alpha}{\pi}\left[\frac{13}{6}\ln\left(\frac{-q^2}{m^2}\right) - \frac{28}{9} - \left[\ln\left(\frac{-q^2}{m^2}\right) - 1\right]\ln\left(\frac{4\epsilon_1\epsilon_3}{(2\eta\Delta E)^2}\right) - \frac{1}{2}\ln^2\eta + L(\cos^2\frac{1}{2}\theta) - \frac{\pi^2}{6}\right] \\
& + \frac{2\alpha Z}{\pi}\left[-\ln\eta\,\ln\left(\frac{-q^2 x}{(2\eta\Delta E)^2}\right) + L\left(1-\frac{\eta}{x}\right) - L\left(1-\frac{1}{\eta x}\right)\right] \\
& + \frac{\alpha Z^2}{\pi}\left[\begin{array}{c} \frac{\epsilon_4}{|\mathbf{p}_4|}\left(-\frac{1}{2}\ln^2 x - \ln x \ln\left(\frac{\rho^2}{M^2}\right) + \ln x\right) - \left(\frac{\epsilon_4}{|\mathbf{p}_4|}\ln x - 1\right)\ln\left(\frac{M^2}{(2\eta\Delta E)^2}\right) + 1 \\ + \frac{\epsilon_4}{|\mathbf{p}_4|}\left(-L\left(1-\frac{1}{x^2}\right) + 2L\left(-\frac{1}{x}\right) + \frac{\pi^2}{6}\right) \end{array}\right] \\
& + \delta_{el}^{(1)}
\end{aligned} \tag{5.2}$$

Here, $\delta_{el}^{(1)}$ is the contribution coming from the inclusion of form factors for the proton and integration over the entire range of four-momenta of the virtual photon in the proton vertex correction (see (3.5), (3.12), (3.13), (3.25)); it is thus not included in the analysis given in [1] and [2], denoted here as the soft photon approximation. Moreover, $\delta_{el}^{(1)}$ has no infrared divergent terms; these are all included in the soft photon approximation.

In Tables I, II, and III we compare the values of the radiative correction, δ, calculated in this paper (denoted by MTj) with those given by Mo and Tsai in [2] for various kinematics. The initial beam energies and momentum transfers have been chosen to correspond to experiments proposed or already performed at Jefferson Lab [11] and SLAC [12]. The final electron detector acceptance, ΔE, has been taken throughout to be one percent of the final electron energy, ϵ_3. In the form factors (see(2.5)), the parameter Λ has been chosen to be 700 MeV/c throughout. The contribution of the terms in (5.2) are grouped according to the power of Z which appears there as a factor. The numerical value of each of these groups of terms is given in the rows denoted by Z^0, Z^1, Z^2. Values given in the column MTj in the row Z^2 do not include the contribution of the proton form factor (which are contained in $\delta_{el}^{(1)}$); they are given for comparison with the values in [2]. In the range of energies and momentum transfers considered here, the correction $\delta_{el}^{(1)}$, due to the finite size of the nucleon (and integration over the entire range of four-momenta of the virtual photon in the proton vertex correction), is found in general to be much smaller than the other contributions with factor Z^2, labeled explicitly in (5.2) and in Tables I, II, and III. The values given in these tables include only the contribution of electron-positron pairs in the vacuum polarization; the contribution of muon and tau pairs is given by (A8) and (A10).

The curves in Figs. 3 and 4 illustrate the two aspects of the present work: (1) the contribution of nucleonic size effects to the radiative correction, and (2) the improvement of the mathematical treatment of the integrations given in the work of Mo and Tsai [1], [2]. The nucleonic size effects are all contained in the term $\delta_{el}^{(1)}$, (Eq.(3.40)); its contribution relative to the overall radiative correction factor, $(1+\delta_{\rm MTj})$, is given by the dashed curves marked VTX, where VTX $= 100\times\delta_{el}^{(1)}/(1+\delta_{\rm MTj})$. The dotted curve shows D0 $= 100\times(\delta_{\rm MTj}^{(0)} - \delta_{\rm Tsai})/(1+\delta_{\rm MTj})$, which is that part of the difference between the radiative correction given by Mo and Tsai [2] and the one given in this paper due solely to the improvement of the mathematical treatment of the integrations. Here, $\delta_{\rm MTj}^{(0)}$ is the radiative correction given in (5.2), *excluding* the term $\delta_{el}^{(1)}$. It will be noticed that VTX is always positive, and that for most of the range of allowed momentum transfers, D0 is negative. Thus their sum, which is the difference between the radiative correction $\delta_{\rm MTj}$ given in this paper in (5.2), and $\delta_{\rm Tsai}$, given in [2], is rather small except for the region corresponding to large scattering angles. This sum is given by the solid curves marked D, where D = D0 + VTX $= 100\times(\delta_{\rm MTj} - \delta_{\rm Tsai})/(1+\delta_{\rm MTj})$.

VI. CONCLUSION

We have calculated the radiative correction to elastic electron-proton scattering to lowest order in α using a hadronic model which includes the finite size of the nucleon. The contribution from the emission of real soft photons by the electron and the proton is calculated exactly. The contributions of the box and crossed-box (two-photon exchange) diagrams are calculated in a soft photon approximation which is less drastic than that employed in [1]. A number of observations may be made from the values given in Tables I, II, and III. First, the contributions of the electron vertex correction, vacuum polarization, and real soft photon emission by the electron (the terms in (5.2) with factor α/π) dominate the radiative correction δ. Since our expression for these terms differs from that given by Mo and Tsai [2] solely in that they have omitted the term $(\alpha/\pi)\left[L(\cos^2\frac{1}{2}\theta) - \frac{\pi^2}{6}\right]$ in (5.2), we find values for δ which differ from theirs by at most 2% for the initial energies and momentum transfers considered here (note that $-\frac{\pi^2}{6} \leq L(\cos^2\frac{1}{2}\theta) - \frac{\pi^2}{6} \leq 0$). Further, we note that, except for the proton and at the higher energies considered here, the contribution of $\delta_{el}^{(1)}$ is negligible. However, for the two highest energies, $\delta_{el}^{(1)}$ is between 2% and 3% of the factor $(1+\delta)$ by which the uncorrected cross section must be multiplied, and hence should be considered in precision measurements for electron-proton scattering at energies above 8 GeV. As an empirical guide, we find that $\delta_{el}^{(1)} = 0.02(1+\delta)$ for initial energies and scattering angles satisfying $\epsilon_1 \sin\theta \approx 8$ for beam energies between 8 and 16 GeV. Finally, we note that a considerable simplification of the expression in (5.2) occurs if, in addition to the last two terms multiplying α/π, we neglect the last two terms multiplying $2\alpha Z/\pi$ as well as the last three terms multiplying $\alpha Z^2/\pi$, each of these sets of terms being always less than $\pi^2/6$ in magnitude. From this study we see that at the energies and momentum transfers considered here, the nucleonic finite size effects are rather small but are expected to become more important at higher energies. The corrections due to the improvement of the high energy behavior of the radiative corrections as described in this paper are not negligible and need to be taken into account at the energies and momentum transfers we have considered.

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APPENDIX A: ELASTIC SCATTERING AMPLITUDES TO ORDER α^2

Using the notation given in section II, the matrix elements corresponding to the various one-loop diagrams shown in Fig. 1 are

$$M_5 = Ze^2\overline{u}(p_3)\Lambda_\mu(p_3,p_1)u(p_1)\frac{(-i)}{q^2+i\epsilon}\overline{u}(p_4)\Gamma^\mu(q^2)u(p_2) \tag{A1}$$

$$M_6 = Z^3e^2\overline{u}(p_3)\gamma_\mu u(p_1)\frac{(-i)}{q^2+i\epsilon}\overline{u}(p_4)\Lambda^\mu(p_4,p_2)u(p_2) \tag{A2}$$

where

$$\Lambda_\mu(p_3,p_1) = -ie^2\int\frac{d^4k}{(2\pi)^4}\frac{1}{k^2-\lambda^2+i\epsilon}\gamma^\nu\frac{1}{(\not{p}_3-\not{k}-m+i\epsilon)}\gamma_\mu\frac{1}{(\not{p}_1-\not{k}-m+i\epsilon)}\gamma_\nu \tag{A3}$$

$$\begin{aligned}\Lambda^\mu(p_4,p_2) = -ie^2\int\frac{d^4k}{(2\pi)^4}\frac{1}{k^2-\lambda^2+i\epsilon}\Gamma^\nu(k^2)\frac{1}{(\not{p}_4-\not{k}-M+i\epsilon)}\Gamma^\mu(q^2)\\ \times\frac{1}{(\not{p}_2-\not{k}-M+i\epsilon)}\Gamma_\nu(k^2)\end{aligned} \tag{A4}$$

$$\begin{aligned}M_2 = (Ze^2)^2\int\frac{d^4k}{(2\pi)^4}&\frac{1}{k^2-\lambda^2+i\epsilon}\,\frac{1}{(k+q)^2-\lambda^2+i\epsilon}\\ &\times\left[\overline{u}(p_3)\gamma_\nu\frac{1}{\not{p}_1-\not{k}-m+i\epsilon}\gamma_\mu u(p_1)\right]\\ &\times\left[\overline{u}(p_4)\Gamma^\nu((k+q)^2)\frac{1}{\not{p}_2+\not{k}-M+i\epsilon}\Gamma^\mu(k^2)u(p_2)\right]\end{aligned} \tag{A5}$$

$$\begin{aligned}M_3 = (Ze^2)^2\int\frac{d^4k}{(2\pi)^4}&\frac{1}{k^2-\lambda^2+i\epsilon}\,\frac{1}{(k+q)^2-\lambda^2+i\epsilon}\\ &\times\left[\overline{u}(p_3)\gamma_\nu\frac{1}{\not{p}_1-\not{k}-m+i\epsilon}\gamma_\mu u(p_1)\right]\\ &\times\left[\overline{u}(p_4)\Gamma^\mu(k^2)\frac{1}{\not{p}_4-\not{k}-M+i\epsilon}\Gamma^\nu((k+q)^2)u(p_2)\right]\end{aligned} \tag{A6}$$

The matrix element, M_4, for vacuum polarization is, after charge renormalization, related simply to the matrix element M_1 given above:

$$M_4 = \Pi(q^2) M_1 \tag{A7}$$

For a fermion loop in the photon propagator, $\Pi(q^2)$ is given in [8] in terms of an integral which can be evaluated in closed form [14], giving

$$\Pi(q^2) = \Pi^f(q^2/m_i^2) = \frac{\alpha}{3\pi}\left\{\left(1 - \frac{u}{2}\right)\sqrt{1+u}\log\left(\frac{\sqrt{1+u}+1}{\sqrt{1+u}-1}\right) + u - \frac{5}{3}\right\} \tag{A8}$$

in which m_i is the mass of the fermion and $u = 4m_i^2/(-q^2)$.

For $-q^2/m_i^2 \gg 1$ this gives

$$\Pi^f(q^2/m_i^2) = \frac{\alpha}{\pi}\left\{\frac{1}{3}\ln\left(\frac{-q^2}{m_i^2}\right) - \frac{5}{9}\right\} \tag{A9}$$

If we include the vacuum polarization amplitudes from particle-antiparticle loops of different masses, as has been done in several experimental analyses [12], then we have

$$M_4 = M_1 \sum_i \Pi^f(q^2/m_i^2) \tag{A10}$$

In principle, once one includes particle-antiparticle pairs of mass greater than the electron mass in the vacuum polarization amplitudes, one should consider bosons as well as fermions. The matrix elements for vacuum polarization for a pair of structureless spin zero bosons in the closed loop, first given by Feynman [13], may be found in a more accessible form in a paper of Tsai [14]. The result, corresponding to the equation above for fermions, is

$$M_4^{boson} = M_1 \sum_i \Pi^b(q^2/m_i^2) \tag{A11}$$

where

$$\Pi^b(q^2/m_i^2) = \frac{\alpha}{3\pi}\left\{\frac{1}{2}\sqrt{1+u}\log\left(\frac{\sqrt{1+u}+1}{\sqrt{1+u}-1}\right) - u - \frac{4}{3}\right\} \tag{A12}$$

A more complete discussion of vacuum polarization should include a consideration of pion structure as well as the contribution of spin-one bosons, in particular the ρ meson. A detailed discussion of the hadronic contribution to vacuum polarization may be found in connection with calculations of the anomalous magnetic moment of the muon [9] and in connection with radiative corrections to high energy electron-positron collider experiments [10].

APPENDIX B: PROTON VERTEX CORRECTION

As noted in Sec. IIIA, the terms $ggg, gsg, ...$ can be expressed in terms of the integrals given in (3.1). There, the integrals I_0, J_0, and K_0 are scalars and hence are functions of the scalars p_2^2, p_4^2,and $p_2 \cdot p_4$ (and, of course, λ^2 and Λ^2). Since we have on-shell particles in the initial and final states ($p_2^2 = p_4^2 = M^2$), these integrals are functions of M^2 and q^2. The integrals I_μ and J_μ are vectors and hence in principal can be written in the form

$$I_\mu = a p_{2\mu} + b p_{4\mu} \tag{B1}$$

with a similar equation for J_μ , where a and b are functions of M^2 and q^2. However, the calculation is simplified greatly if we express I_μ in terms of the four-vectors $\rho = p_4 + p_2$ (which is symmetric in p_4 and p_2) and $q = p_4 - p_2$ (which is antisymmetric in p_4 and p_2), *i.e.* $I_\mu = A\rho_\mu + Bq_\mu$. Here A and B are functions of M^2 and q^2 and hence are symmetric in p_4 and p_2. Further, since the integrands for the vectors I_μ and J_μ are symmetric in p_4 and p_2, it follows that $B = 0$. We thus have

$$I_\mu = A\rho_\mu \tag{B2}$$

and a similar equation for J_μ. These same considerations of symmetry allow for the simplification of the tensors $I_{\mu\nu}$ and $J_{\mu\nu}$, which are also symmetric functions of p_4 and p_2. We can therefore write

$$I_{\mu\nu} = a_1 \rho_\mu \rho_\nu + a_2 q_\mu q_\nu + a_3 g_{\mu\nu} \tag{B3}$$

and a similar equation for $J_{\mu\nu}$. That the terms $\rho_\mu q_\nu$ and $q_\mu \rho_\nu$ are absent follows directly by multiplying $I_{\mu\nu}$ successively by $\rho^\mu q^\nu$ and $q^\mu \rho^\nu$, using $\rho \cdot q = 0$ and the fact that $I_{\mu\nu}\rho^\mu q^\nu$ and $I_{\mu\nu} q^\mu \rho^\nu$ are antisymmetric in p_2 and p_4. Multiplying I_μ (J_μ) by ρ^μ, and $I_{\mu\nu}$ ($J_{\mu\nu}$) successively by $\rho^\mu\rho^\nu, q^\mu q^\nu$, and $g^{\mu\nu}$, the coefficients in the expressions for $I_\mu, J_\mu, I_{\mu\nu}$ and $J_{\mu\nu}$ may be expressed in terms of their moments, defined by

$$g_1 = \frac{1}{\rho^2} I_\mu \rho^\mu, \qquad h_1 = \frac{1}{\rho^2} J_\mu \rho^\mu \tag{B4}$$

$$g_{11} = \frac{1}{\rho^4} I_{\mu\nu} \rho^\mu \rho^\nu, \qquad h_{11} = \frac{1}{\rho^4} J_{\mu\nu} \rho^\mu \rho^\nu \tag{B5}$$

$$g_{22} = \frac{1}{\rho^4} I_{\mu\nu} q^\mu q^\nu, \qquad h_{22} = \frac{1}{\rho^4} J_{\mu\nu} q^\mu q^\nu \tag{B6}$$

Conversely, the integrals in (3.1) may be written in terms of the moments:

$$I_\mu = \rho_\mu g_1, \qquad J_\mu = \rho_\mu h_1 \tag{B7}$$

$$\begin{aligned} I_{\mu\nu} = \rho_\mu \rho_\nu & \left[g_{11} - \frac{1}{2\rho^2}(h_0 - \rho^2 g_{11} - q^2 g_{22}) \right] \\ & + q_\mu q_\nu \left[g_{22} - \frac{1}{2q^2}(h_0 - \rho^2 g_{11} - q^2 g_{22}) \right] \\ & + g_{\mu\nu} \left[\frac{1}{2}(h_0 - \rho^2 g_{11} - q^2 g_{22}) \right] \end{aligned} \tag{B8}$$

$$J_{\mu\nu} = \rho_\mu\rho_\nu \left[h_{11} - \frac{1}{2\rho^2}(k_0 - \rho^2 h_{11} - q^2 h_{22})\right] + q_\mu q_\nu \left[h_{22} - \frac{1}{2q^2}(k_0 - \rho^2 h_{11} - q^2 h_{22})\right] + g_{\mu\nu}\left[\frac{1}{2}(k_0 - \rho^2 h_{11} - q^2 h_{22})\right] \tag{B9}$$

where, for convenience of notation, we have defined

$$g_0 = I_0, \qquad h_0 = J_0, \qquad k_0 = K_0 \tag{B10}$$

The terms $ggg, gsg, ...$ can be expressed in terms of these moments. Substituting (2.4) and (2.5) in the expression for the proton vertex correction, Eq. (A4), and substituting this in turn in the matrix element M_6, Eq. (A2), the integrals are all of the form given in Eq. (3.1), which are in turn expressed in terms of the moments by Eqs. (B7)-(B10). We make use of the fact that in M_6 the vertex correction is taken between free spinors, and finally express p_2 and p_4 in terms of ρ and q. We find

$$ggg = -ie^2 F(q^2)\left\{ \begin{array}{c} \left[\begin{array}{c} 2(2M^2-q^2)g_0 - 4(2M^2-q^2)g_1 \\ -2(8M^2+q^2)g_{11} - 2(q^4/\rho^2)g_{22} + 8(M^2/\rho^2)h_0 \end{array}\right]\gamma_\mu \\ + \left[-8M^2 g_1 + 24M^2 g_{11} + 8M^2(q^2/\rho^2)g_{22} - 8(M^2/\rho^2)h_0\right]\frac{i\sigma_{\mu\nu}q^\nu}{2M} \end{array}\right\} \tag{B11}$$

$$gsg = -ie^2\kappa F(q^2)\left\{ \begin{array}{c} \left[-2q^2 g_1\right]\gamma_\mu \\ + \left[2(2M^2-q^2)g_0 - 4(2M^2-q^2)g_1\right]\frac{i\sigma_{\mu\nu}q^\nu}{2M} \end{array}\right\} \tag{B12}$$

$$ggs + sgg = -2ie^2\kappa F(q^2)\left\{ \begin{array}{c} \left[-12M^2 g_{11} - (q^4/\rho^2)g_{22} + 2(1+2M^2/\rho^2)h_0 - 3h_1\right]\gamma_\mu \\ + \left[\begin{array}{c} (16M^2-q^2)g_{11} + (q^2/\rho^2)(8M^2-q^2)g_{22} \\ -4(1+M^2/\rho^2)h_0 + 3h_1 \end{array}\right]\frac{i\sigma_{\mu\nu}q^\nu}{2M} \end{array}\right\} \tag{B13}$$

$$gss + ssg = -2ie^2\kappa^2 F(q^2)\left\{ \begin{array}{c} (q^2/4M^2)\left[\begin{array}{c} -(8M^2+q^2)g_{11} - (q^4/\rho^2)g_{22} \\ +(-2+4M^2/\rho^2)h_0 + h_1 \end{array}\right]\gamma_\mu \\ + \left[3q^2 g_{11} + 4M^2(q^2/\rho^2)g_{22} - (q^2/\rho^2)h_0 - h_1\right]\frac{i\sigma_{\mu\nu}q^\nu}{2M} \end{array}\right\} \tag{B14}$$

$$sgs = -ie^2\kappa^2 F(q^2)\left\{ \begin{array}{c} \left[\begin{array}{c} -(8M^2+q^2)g_{11} - (q^4/\rho^2)g_{22} + (2+q^2/\rho^2)h_0 \\ -(8M^2+q^2)h_{11}/4M^2 - (q^4/\rho^2)h_{22}/4M^2 + (-1+q^2/\rho^2)k_0/4M^2 \end{array}\right]\gamma_\mu \\ + \left[\begin{array}{c} 12M^2 g_{11} + 4M^2(q^2/\rho^2)g_{22} - 2(1+2M^2/\rho^2)h_0 \\ +3h_{11} + 4(q^2/\rho^2)h_{22} - k_0/\rho^2 \end{array}\right]\frac{i\sigma_{\mu\nu}q^\nu}{2M} \end{array}\right\} \tag{B15}$$

$$sss = -ie^2\kappa^3 F(q^2)\left\{ \begin{array}{c} (q^2/4M^2)\left[\begin{array}{c} -12M^2 g_{11} - 4M^2(q^2/\rho^2)g_{22} + 2(1+2M^2/\rho^2)h_0 \\ -4h_1 + 3h_{11} + (q^2/\rho^2)h_{22} - k_0/\rho^2 \end{array}\right]\gamma_\mu \\ + \left[\begin{array}{c} 2(2M^2+q^2)g_{11} + 4M^2(q^2/\rho^2)g_{22} \\ -(4M^2/\rho^2 + q^2/2M^2)h_0 + (q^2/M^2)h_1 \\ -2(2M^2+q^2)h_{11}/4M^2 + 2(q^2/\rho^2)(2M^2-q^2)h_{22}/4M^2 \\ +(q^2/\rho^2)k_0/4M^2 \end{array}\right]\frac{i\sigma_{\mu\nu}q^\nu}{2M} \end{array}\right\} \tag{B16}$$

The expressions do not depend on the particular form of the form factors; we have assumed only that $F_1 = F_2 = F$. However, for form factors of the form given in Eq. (2.5), the moments may all be expressed more simply in terms of the functions $C(\Lambda^2)$:

$$\left\{C_0(\Lambda^2); C_\mu(\Lambda^2); C_{\mu\nu}(\Lambda^2)\right\} = \int d^4k \, \{1; k_\mu; k_{\mu\nu}\} / D(\Lambda^2) \tag{B17}$$

Using the identity

$$\frac{1}{k^2 - \lambda^2}\left(\frac{-\Lambda^2}{k^2 - \Lambda^2}\right)^m = \frac{(-\Lambda^2)^m}{(m-1)!} T^{m-1}\left\{\frac{1}{\Lambda^2 - \lambda^2}\left[\frac{1}{k^2 - \Lambda^2} - \frac{1}{k^2 - \lambda^2}\right]\right\} \tag{B18}$$

with $T \equiv \frac{\partial}{\partial(\Lambda^2)}$, we can write

$$\{I\} = N_m^{'}(\Lambda^2)^m T^{m-1}\left\{\frac{C(\Lambda^2) - C(\lambda^2)}{\Lambda^2 - \lambda^2}\right\}, \tag{B19}$$

where

$$N_m^{'} = \frac{(-1)^m}{(m-1)!(2\pi)^4}. \tag{B20}$$

In Eq. (B19) I and C denote any one of $I_0, I_\mu, I_{\mu\nu}$ and $C_0, C_\mu, C_{\mu\nu}$ respectively. We see from Eq. (B19) that terms in $C(\Lambda^2)$ which are independent of Λ^2 do not appear in the expression for I . In particular, this applies to $C_{\mu\nu}(\Lambda^2)$, which may be evaluated using either dimensional regularization or a convergence factor. The infinities in $C_{\mu\nu}(\Lambda^2)$ are indeed independent of Λ^2, thus giving a finite result for $I_{\mu\nu}$ as it should. In similar fashion, we have

$$\{J\} = N_m^{'}(\Lambda^2)^m T^{m-1}\left\{\frac{\Lambda^2 C(\Lambda^2) - \lambda^2 C(\lambda^2)}{\Lambda^2 - \lambda^2}\right\} \tag{B21}$$

in which J and C denote any one of $J_0, J_\mu, J_{\mu\nu}$ and $C_0, C_\mu, C_{\mu\nu}$ respectively. We see from Eq. (B21) that any terms in $C(\Lambda^2)$ which are independent of Λ^2 do not appear in the expression for J provided $m > 1$. Finally, for K_0 we get

$$K_0 = N_m^{'}(\Lambda^2)^m T^{m-1}\left\{\frac{\Lambda^4 C_0(\Lambda^2) - \lambda^4 C_0(\lambda^2)}{\Lambda^2 - \lambda^2}\right\} \tag{B22}$$

We note that, apart from trivial factors, the integrals in (B17) are the three-point functions defined in [3], Eq.(5.1), and [4], Eq.(E.1); C_0 has been evaluated in terms of Spence functions in [3]. The details of the algebra in [3] and [4] being rather lengthy, we choose instead to evaluate the integrals in (B17) using Feynman parameters, writing

$$\frac{1}{D(\Lambda^2)} = 2\int_0^1 \int_0^1 \frac{x\,dx\,dy}{[k^2 - 2xk\cdot p_y - \Lambda^2(1-x) + i\epsilon]^3}, \tag{B23}$$

where $p_y = p_2 y + p_4(1-y)$. Substituting Eq. (B23) in Eq. (B17) and shifting the integration variable k ($k - xp_y \to k$) we then have

$$\begin{aligned}&\left\{C_0(\Lambda^2); C_\mu(\Lambda^2); C_{\mu\nu}(\Lambda^2)\right\}\\ &= -i\pi^2\int_0^1\int_0^1 dx\, dy \frac{\left\{x; x^2 p_{y\mu}; x^3 p_{y\mu}p_{y\nu} + \frac{1}{2}g_{\mu\nu}\lambda^2(x - \frac{1}{2}x^2)\right\}}{x^2 p_y^2 + \Lambda^2(1-x)} \;+\; \chi_{\mu\nu},\end{aligned} \tag{B24}$$

where, throughout this section, we denote by $\chi_{\mu\nu}$ any terms which are independent of Λ . Rewriting $p_y = \frac{1}{2}\rho + \frac{1}{2}q(1-2y)$ in Eqs. (B23) and (B24) we may, neglecting terms which are independent of Λ, express C_0, C_μ and $C_{\mu\nu}$ in terms of the functions

$$\phi_k(\Lambda^2) \equiv \int_0^1\int_0^1 \frac{x^k\, dx\, dy}{p_y^2 x^2 + \Lambda^2(1-x)}. \tag{B25}$$

We get

$$C_0 = -i\pi^2\phi_1(\lambda^2);\;\; C_\mu = -i\pi^2\frac{1}{2}\rho_\mu\phi_2(\lambda^2) \tag{B26}$$

$$\begin{aligned}C_{\mu\nu} = -i\pi^2 \Bigg[&\frac{1}{4}\rho_\mu\rho_\nu\phi_3(\lambda^2) - \frac{1}{4}\frac{\rho^2}{q^2}q_\mu q_\nu\phi_3(\lambda^2) + \\ &-\frac{q_\mu q_\nu}{q^2}\lambda^2[\phi_1(\lambda^2) - \phi_2(\lambda^2)] + \frac{1}{2}g_{\mu\nu}\lambda^2[\phi_1(\lambda^2) - \frac{1}{2}\phi_2(\lambda^2)]\Bigg].\end{aligned} \tag{B27}$$

As shown in Appendix D, the functions ϕ_k obey a three-term inhomogeneous recursion, which is used to calculate ϕ_k for $k > 1$:

$$\begin{aligned}&(k+1)\rho^2\phi_{k+2}(\Lambda^2) - 2(2k+1)\Lambda^2\phi_{k+1}(\Lambda^2) + 4k\Lambda^2\phi_k(\Lambda^2)\\ &\qquad = \frac{2\rho}{\rho_1}\ln\left(\frac{\rho+\rho_1}{\rho-\rho_1}\right) + 2\Lambda^2\left[\phi_{k+1}^{(0)}(\Lambda^2) - 2\phi_k^{(0)}(\Lambda^2)\right]\end{aligned} \tag{B28}$$

Here

$$\phi_k^{(0)}(\Lambda^2) \equiv \phi_k(\Lambda^2)|_{-q^2=0} = \int_0^1 \frac{x^k\, dx}{M^2x^2 + (1-x)\Lambda^2} \tag{B29}$$

The functions $\phi_k^{(0)}(\Lambda^2)$ may in turn be calculated from the recursion

$$M^2\phi_{k+2}^{(0)}(\Lambda^2) - \Lambda^2\phi_{k+1}^{(0)}(\Lambda^2) + \Lambda^2\phi_k^{(0)}(\Lambda^2) = \frac{1}{k+1} \tag{B30}$$

To implement the recursions (B28) and (B30) we need

$$\phi_0^{(0)}(\Lambda^2) = \frac{1}{\Lambda\Lambda_1}\ln\left(\frac{\Lambda+\Lambda_1}{\Lambda-\Lambda_1}\right) \tag{B31}$$

and

$$\phi_1^{(0)}(\Lambda^2) = \frac{1}{2M^2}\left[\ln\frac{M^2}{\Lambda^2} + \frac{\Lambda}{\Lambda_1}\ln\left(\frac{\Lambda+\Lambda_1}{\Lambda-\Lambda_1}\right)\right] \tag{B32}$$

which follow from Eq. (B29), and $\phi_1(\Lambda^2)$, which can be expressed in terms of dilogarithms (Spence functions)(see Appendix D):

$$\phi_1(\Lambda^2) = \frac{1}{\rho\rho_1}\left\{L\left(1-\frac{1}{xy}\right) - L\left(1-\frac{x}{y}\right) - 2\ln(x)\ln(1+\frac{1}{y})\right\} \tag{B33}$$

where

$$L(z) = -\int_0^z \frac{\ln(1-t)}{t}\,dt \tag{B34}$$

$$x = \frac{\rho+\rho_1}{\rho-\rho_1} = \frac{(\rho+\rho_1)^2}{4M^2} \tag{B35}$$

$$y = \frac{\Lambda+\Lambda_1}{\Lambda-\Lambda_1} = \frac{(\Lambda+\Lambda_1)^2}{4M^2} \tag{B36}$$

We will also want to take the limit $\lambda \to 0$. Neglecting all terms which vanish in this limit, we find,

$$\phi_1(\lambda^2) \xrightarrow[\lambda\to 0]{} \frac{1}{\rho\rho_1}\left\{-2L\left(-\frac{1}{x}\right) - \frac{\pi^2}{6} - \frac{1}{2}\ln^2 x + \ln x\,\ln\left(\frac{\rho^2}{\lambda^2}\right)\right\} \tag{B37}$$

$$\phi_1^{(0)}(\lambda^2) \xrightarrow[\lambda\to 0]{} \frac{1}{M^2}\ln\left(\frac{M}{\lambda}\right) \tag{B38}$$

and for $k > 1$,

$$\phi_k(0) = \frac{2}{(k-1)\rho\rho_1}\ln x \tag{B39}$$

$$\phi_k^{(0)}(0) = \frac{1}{(k-1)M^2} \tag{B40}$$

We then find

$$g_0 = -N_1\phi_1(\lambda^2) + N_m(\Lambda^2)^m T^{m-1}\left\{\frac{1}{\Lambda^2}\phi_1(\Lambda^2)\right\} \tag{B41}$$

$$g_1 = \frac{1}{2}N_m(\Lambda^2)^m T^{m-1}\left\{\frac{1}{\Lambda^2}\left[\phi_2(\Lambda^2)-\phi_2(0)\right]\right\} \tag{B42}$$

$$\begin{aligned} g_{11} = {} & \frac{1}{4}N_m(\Lambda^2)^m T^{m-1}\left\{\frac{1}{\Lambda^2}\left[\phi_3(\Lambda^2)-\phi_3(0)\right]\right\} \\ & + \frac{1}{2\rho^2}N_m(\Lambda^2)^m T^{m-1}\left\{\phi_1(\Lambda^2) - \frac{1}{2}\phi_2(\Lambda^2)\right\} \end{aligned} \tag{B43}$$

$$g_{22} = -\frac{\rho^2}{4q^2} N_m (\Lambda^2)^m T^{m-1} \left\{ \frac{1}{\Lambda^2} \left[\phi_3(\Lambda^2) - \phi_3(0) \right] \right\} \tag{B44}$$
$$-\frac{1}{2q^2} N_m (\Lambda^2)^m T^{m-1} \left\{ \phi_1(\Lambda^2) - \frac{3}{2}\phi_2(\Lambda^2) \right\}$$

$$h_0 = N_m (\Lambda^2)^m T^{m-1} \left\{ \phi_1(\Lambda^2) \right\} \tag{B45}$$

$$h_1 = \frac{1}{2} N_m (\Lambda^2)^m T^{m-1} \left\{ \phi_2(\Lambda^2) \right\} \tag{B46}$$

$$h_{11} = \frac{1}{4} N_m (\Lambda^2)^m T^{m-1} \left\{ \phi_3(\Lambda^2) \right\} \tag{B47}$$
$$+\frac{1}{2\rho^2} N_m (\Lambda^2)^m T^{m-1} \left\{ \Lambda^2 \left[\phi_1(\Lambda^2) - \frac{1}{2}\phi_2(\Lambda^2) \right] \right\}$$

$$h_{22} = -\frac{\rho^2}{4q^2} N_m (\Lambda^2)^m T^{m-1} \left\{ \phi_3(\Lambda^2) \right\} \tag{B48}$$
$$-\frac{1}{2q^2} N_m (\Lambda^2)^m T^{m-1} \left\{ \Lambda^2 \left[\phi_1(\Lambda^2) - \frac{3}{2}\phi_2(\Lambda^2) \right] \right\}$$

$$k_0 = N_m (\Lambda^2)^m T^{m-1} \left\{ \Lambda^2 \phi_1(\Lambda^2) \right\} \tag{B49}$$

where

$$N_m = -i\pi^2 N_m^{'} \tag{B50}$$

The terms $ggg, gsg, ...sss$ can now be expressed more simply in terms of the functions ϕ_k. We get

$$ggg = -ie^2 F(q^2) \left\{ \begin{array}{c} -2(2M^2 - q^2) N_1 \phi_1(\lambda^2) \\ +2(2M^2 - q^2) \left[S^{m-1} \left\{ \frac{1}{\Lambda^2} \phi_1(\Lambda^2) \right\} - S^{m-1} \left\{ \frac{1}{\Lambda^2} \left[\phi_2(\Lambda^2) - \phi_2(0) \right] \right\} \right] \\ +S^{m-1} \left\{ \phi_2(\Lambda^2) \right\} - 4M^2 S^{m-1} \left\{ \frac{1}{\Lambda^2} \left[\phi_3(\Lambda^2) - \phi_3(0) \right] \right\} \end{array} \right\} \gamma_\mu \tag{B51}$$
$$-ie^2 F(q^2) \left\{ -4M^2 S^{m-1} \left\{ \frac{1}{\Lambda^2} \left[\phi_2(\Lambda^2) - \phi_2(0) \right] \right\} + 4M^2 S^{m-1} \left\{ \frac{1}{\Lambda^2} \left[\phi_3(\Lambda^2) - \phi_3(0) \right] \right\} \right\} \frac{i\sigma_{\mu\nu} q^\nu}{2M}$$

$$gsg = -ie^2 \kappa F(q^2) \left\{ -q^2 S^{m-1} \left\{ \frac{1}{\Lambda^2} \left[\phi_2(\Lambda^2) - \phi_2(0) \right] \right\} \right\} \gamma_\mu \tag{B52}$$
$$-ie^2 \kappa F(q^2) \left\{ 2(2M^2 - q^2) \left[\begin{array}{c} -N_1 \phi_1(\lambda^2) + S^{m-1} \left\{ \frac{1}{\Lambda^2} \phi_1(\Lambda^2) \right\} \\ -S^{m-1} \left\{ \frac{1}{\Lambda^2} \left[\phi_2(\Lambda^2) - \phi_2(0) \right] \right\} \end{array} \right] \right\} \frac{i\sigma_{\mu\nu} q^\nu}{2M}$$

$$ggs + sgg = -2ie^2 \kappa F(q^2) \left\{ \begin{array}{c} (\frac{q^2}{4} - 3M^2) S^{m-1} \left\{ \frac{1}{\Lambda^2} \left[\phi_3(\Lambda^2) - \phi_3(0) \right] \right\} \\ +\frac{3}{2} S^{m-1} \left\{ \phi_1(\Lambda^2) - \frac{1}{2}\phi_2(\Lambda^2) \right\} \end{array} \right\} \gamma_\mu \tag{B53}$$
$$+ - 2ie^2 \kappa F(q^2) \left\{ \begin{array}{c} 2M^2 S^{m-1} \left\{ \frac{1}{\Lambda^2} \left[\phi_3(\Lambda^2) - \phi_3(0) \right] \right\} \\ -4S^{m-1} \left\{ \phi_1(\Lambda^2) - \frac{1}{2}\phi_2(\Lambda^2) \right\} \end{array} \right\} \frac{i\sigma_{\mu\nu} q^\nu}{2M}$$

$$gss + ssg = -2ie^2\kappa^2 F(q^2)\frac{q^2}{4M^2}\left\{\begin{array}{c} -2M^2 S^{m-1}\left\{\frac{1}{\Lambda^2}\left[\phi_3(\Lambda^2)-\phi_3(0)\right]\right\} \\ -2S^{m-1}\left\{\phi_1(\Lambda^2)-\frac{1}{2}\phi_2(\Lambda^2)\right\}\end{array}\right\}\gamma_\mu \tag{B54}$$

$$+ - 2ie^2\kappa^2 F(q^2)\left\{\begin{array}{c} (\frac{3}{4}q^2 - M^2)S^{m-1}\left\{\frac{1}{\Lambda^2}\left[\phi_3(\Lambda^2)-\phi_3(0)\right]\right\} \\ -\frac{1}{2}S^{m-1}\left\{\phi_1(\Lambda^2)-\frac{1}{2}\phi_2(\Lambda^2)\right\}\end{array}\right\}\frac{i\sigma_{\mu\nu}q^\nu}{2M}$$

$$sgs = -ie^2\kappa^2 F(q^2)\left\{\begin{array}{c} -2M^2 S^{m-1}\left\{\frac{1}{\Lambda^2}\left[\phi_3(\Lambda^2)-\phi_3(0)\right]\right\} - \frac{1}{2}S^{m-1}\left\{\phi_3(\Lambda^2)\right\} \\ +S^{m-1}\left\{\phi_1(\Lambda^2)\right\} + \frac{1}{2}S^{m-1}\left\{\phi_2(\Lambda^2)\right\} \\ -\frac{1}{2M^2}S^{m-1}\left\{\Lambda^2\left[\phi_1(\Lambda^2)-\frac{1}{4}\phi_2(\Lambda^2)\right]\right\}\end{array}\right\}\gamma_\mu \tag{B55}$$

$$-ie^2\kappa^2 F(q^2)\left\{\begin{array}{c} 2M^2 S^{m-1}\left\{\frac{1}{\Lambda^2}\left[\phi_3(\Lambda^2)-\phi_3(0)\right]\right\} \\ -2S^{m-1}\left\{\phi_1(\Lambda^2)\right\} + \frac{1}{2}S^{m-1}\left\{\phi_3(\Lambda^2)\right\}\end{array}\right\}\frac{i\sigma_{\mu\nu}q^\nu}{2M}$$

$$sss = -ie^2\kappa^3 F(q^2)\frac{q^2}{4M^2}\left\{\begin{array}{c} -2M^2 S^{m-1}\left\{\frac{1}{\Lambda^2}\left[\phi_3(\Lambda^2)-\phi_3(0)\right]\right\} + \frac{1}{2}S^{m-1}\left\{\phi_3(\Lambda^2)\right\} \\ +2S^{m-1}\left\{\phi_1(\Lambda^2)-\phi_2(\Lambda^2)\right\}\end{array}\right\}\gamma_\mu \tag{B56}$$

$$-ie^2\kappa^3 F(q^2)\left\{\begin{array}{c} \frac{1}{2}q^2 S^{m-1}\left\{\frac{1}{\Lambda^2}\left[\phi_3(\Lambda^2)-\phi_3(0)\right]\right\} - (1+\frac{q^2}{2M^2})S^{m-1}\left\{\phi_1(\Lambda^2)\right\} \\ -\frac{1}{2}S^{m-1}\left\{\phi_3(\Lambda^2)\right\} \\ +\frac{1}{2}(1+\frac{q^2}{M^2})S^{m-1}\left\{\phi_2(\Lambda^2)\right\} - \frac{1}{4M^2}S^{m-1}\left\{\Lambda^2\left[\phi_1(\Lambda^2)-\phi_2(\Lambda^2)\right]\right\}\end{array}\right\}\frac{i\sigma_{\mu\nu}q^\nu}{2M}$$

where

$$S^{m-1} = N_m(\Lambda^2)^m T^{m-1} \tag{B57}$$

It should be noted that the terms with $\phi_1(\lambda^2)$, which appear only in ggg and gsg, constitute the well-known infrared divergence. They are, apart from the hard-photon proton interaction, (2.4), independent of the proton form factor (in this case independent of Λ and M). This is to be expected, since this term is cancelled by a similar infrared divergent term coming from the cross section for the emission of a real soft photon, which is given by the elastic cross section multiplied by a factor independent of the proton form factor.

APPENDIX C: ELECTRON VERTEX CORRECTION

As noted in Sec. IIIC, the electron vertex correction, M_5, may be obtained directly from the proton vertex correction, M_6, if we retain only the term ggg, set $F = 1$, replace p_2, p_4 and M by p_1, p_3 and m and take the limit $\Lambda \to \infty$. After making these replacements in ggg as given in (B51), we need $\phi_k(\Lambda^2)$ to order m^2/Λ^2 and $\phi_k(0)$ (see (B39)). From (B33), writing ([3], p. 389 (B.2))

$$L(1-z) = -L(z) + \frac{1}{6}\pi^2 - \ln z\,\ln(1-z)$$

and neglecting terms of relative order m^2/Λ^2, we have

$$\phi_1(\Lambda^2) \xrightarrow[\Lambda\to\infty]{} \frac{1}{\Lambda^2}\left\{1 + \ln\left(\frac{\Lambda^2}{m^2}\right) - \frac{\rho_m}{\rho_1}\ln x_m\right\} \tag{C1}$$

and from (B25)

$$\phi_k(\Lambda^2) - \phi_{k+1}(\Lambda^2) \xrightarrow[\Lambda \to \infty]{} \frac{1}{\Lambda^2} \left\{ \frac{1}{k+1} \right\} \tag{C2}$$

Choosing $m = 1$ in (B51) we then have

$$ggg = \left[G_1^{(e)}(q^2)\gamma_\mu + G_2^{(e)}(q^2)\frac{i\sigma_{\mu\nu}q^\nu}{2m} \right] \tag{C3}$$

where

$$G_1^{(e)}(q^2) = \frac{\alpha}{4\pi} \left\{ -2(2m^2 - q^2)\phi_1(\lambda^2) + \left(\frac{3\rho_m^2 - 4m^2}{\rho_m \rho_1} \right) \ln x_m + \frac{1}{2} + \ln \left(\frac{\Lambda^2}{m^2} \right) \right\} \tag{C4}$$

$$G_2^{(e)}(q^2) = \frac{\alpha}{4\pi} \left\{ \frac{4m^2}{\rho_m \rho_1} \ln x_m \right\} \tag{C5}$$

Adding the contribution of the electron self energy diagrams gives

$$\overline{ggg} = \left[\left(G_1^{(e)}(q^2) - G_1^{(e)}(0) \right) \gamma_\mu + G_2^{(e)}(q^2)\frac{i\sigma_{\mu\nu}q^\nu}{2m} \right] \tag{C6}$$

To facilitate comparison with [1], we write this in terms of the functions $K(p_i, p_j)$. Similarly to (3.7), we now have

$$(2m^2 - q^2)\phi_1(\lambda^2) = K(p_1, p_3) \tag{C7}$$

which then gives (3.18).

APPENDIX D: THE FUNCTIONS $\phi_k\left(\Lambda^2\right)$

In this Appendix we derive the three-term recurrence relation for the function $\phi_k(\Lambda^2)$ given in (B28) as well as the expression for the function $\phi_1(\Lambda^2)$, defined in (B25) and given in terms of Spence functions in (B33). Integrating first over y in Eq. (B25) we have

$$\phi_k(\Lambda^2) = \frac{1}{\rho_1} \int_0^1 \frac{x^{k-1}}{R} \log \left\{ \frac{R + x\rho_1}{R - x\rho_1} \right\} dx, \tag{D1}$$

where $R^2 = \rho^2 x^2 + 4(1-x)\Lambda^2$ and $\rho_1^2 = -q^2 > 0$. Noting that

$$\frac{d}{dx}\left\{x^k R\right\} = x^{k-1}\left\{kR^2 + \rho^2 x^2 - 2x\Lambda^2\right\} R^{-1}, \tag{D2}$$

we get

$$(k+1)\rho^2 \phi_{k+2} - 2(2k+1)\Lambda^2 \phi_{k+1} + 4k\Lambda^2 \phi_k = \frac{2}{\rho_1} \int_0^1 \log \left\{ \frac{R + x\rho_1}{R - x\rho_1} \right\} d\left(x^k R\right). \tag{D3}$$

Integration by parts then gives

$$(k+1)\rho^2\phi_{k+2} - 2(2k+1)\Lambda^2\phi_{k+1} + 4k\Lambda^2\phi_k = \frac{2\rho}{\rho_1}\log\left(\frac{\rho+\rho_1}{\rho-\rho_1}\right) - 2\Lambda^2\int_0^1 \frac{x^k(2-x)}{M^2x^2+(1-x)\Lambda^2} \tag{D4}$$

from which the recursion (B28) follows at once, using Eq. (B29)

From Eq. (D4) it is clear that ϕ_2 and ϕ_3 follow once we have evaluated ϕ_1, which follows. Setting $k=1$ in (B25) and integrating first over x, we get

$$\int_0^1 \frac{x\,dx}{p_y^2x^2+\Lambda^2(1-x)} = \frac{1}{2p_y^2}\left[\ln\left(\frac{p_y^2}{\Lambda^2}\right) + \frac{\Lambda}{\Delta}\ln\left(\frac{\Lambda+\Delta}{\Lambda-\Delta}\right)\right], \tag{D5}$$

where $\Delta^2 = \Lambda^2 - 4p_y^2$. We next make the change of variable $y = (1+\omega)/2$, which gives $\Delta^2 = \rho_1^2\omega^2 + \Lambda^2 - \rho^2$, and then make the further change of variable $\Delta = \rho_1\omega + s$, from which

$$\omega = \frac{\Lambda^2-\rho^2-s^2}{2\rho_1 s}; \qquad \Delta = \frac{\Lambda^2-\rho^2+s^2}{2s} \tag{D6}$$

Then integrating (B25) over y gives

$$\begin{aligned}\phi_1(\Lambda^2) = &\frac{2}{\rho_1}\int_{s_-}^{s_+}\frac{ds}{\rho^2-(\Lambda-s)^2}\ln\left[\frac{(\Lambda+s)^2-\rho^2}{4s\Lambda}\right] \\ &-\frac{2}{\rho_1}\int_{s_-}^{s_+}\frac{ds}{(\Lambda+s)^2-\rho^2}\ln\left[\frac{\rho^2-(\Lambda-s)^2}{4s\Lambda}\right]\end{aligned} \tag{D7}$$

where

$$s_\pm = \Lambda_1 \pm \rho_1 \tag{D8}$$

Factoring the expressions which appear as factors to the logarithms as well as in their arguments, we can write

$$\phi_1(\Lambda^2) = \frac{1}{\rho\rho_1}\sum_{i=1}^{9} I_i \tag{D9}$$

where

$$\begin{aligned}&I_1 = \int_{s_-}^{s_+}\frac{ds}{s-\sigma_-}\ln(s+\sigma_+);\ I_2 = \int_{s_-}^{s_+}\frac{ds}{s+\sigma_+}\ln(s-\sigma_-)\\ &I_3 = \int_{s_-}^{s_+}\frac{ds}{\sigma_+-s}\ln(s+\sigma_-);\ I_4 = -\int_{s_-}^{s_+}\frac{ds}{s+\sigma_-}\ln(\sigma_+-s)\\ &I_5 = \int_{s_-}^{s_+}\frac{ds}{s-\sigma_-}\ln\left(\frac{s+\sigma_-}{2s}\right);\ I_8 = -\int_{s_-}^{s_+}\frac{ds}{s+\sigma_-}\ln\left(\frac{s-\sigma_-}{2s}\right)\end{aligned} \tag{D10}$$

$$I_6 = \int_{s_-}^{s_+} \frac{ds}{s+\sigma_+} \ln\left(\frac{\sigma_+ - s}{2s}\right); \; I_7 = \int_{s_-}^{s_+} \frac{ds}{\sigma_+ - s} \ln\left(\frac{s+\sigma_+}{2s}\right)$$

$$I_9 = -\int_{s_-}^{s_+} ds \left[\frac{1}{s-\sigma_-} + \frac{1}{\sigma_+ - s} - \frac{1}{\bar{s}+\sigma_-} + \frac{1}{s+\sigma_+}\right] \ln(2\Lambda)$$

where

$$\sigma_\pm = \Lambda \pm \rho \tag{D11}$$

Some of the integrals I_i can be integrated out directly. For these terms we get:

$$\sum_{n=1}^{4} I_n = \ln\left(\frac{\alpha_-}{\alpha_+}\right) ln\left(\frac{4\sigma_+\sigma_-}{(1-\alpha_+^2)(1-\alpha_-^2)}\right) - \ln\alpha_+ \ln(1-\alpha_+^2) + \ln\alpha_- \ln(1-\alpha_-^2) \tag{D12}$$

and

$$I_9 = -2\ln\left(\frac{\alpha_-}{\alpha_+}\right) \ln(2\Lambda) \tag{D13}$$

where

$$\alpha_\pm = \frac{\rho \mp \rho_1}{\Lambda + \Lambda_1}$$

The remaining terms in Eq. (D9) we may rewrite as

$$\sum_{n=5}^{8} I_n = \ln\alpha_+ \ln(1-\alpha_+^2) - \ln\alpha_- \ln(1-\alpha_-^2) + L(1-\alpha_+^2) - L(1-\alpha_-^2) \tag{D14}$$

where L is the dilogarithm (Spence) function. We then get the result given in (B33) in Appendix B.

APPENDIX E: FINAL ELECTRON DETECTOR ACCEPTANCE

In this Appendix, we express $\Delta\epsilon$, the maximum momentum of the photon in the frame S^0, in terms of the final electron detector acceptance in the lab frame, ΔE. In S^0 ($\mathbf{p}_4 + \mathbf{k} = 0$), if $|\mathbf{k}| = \Delta\epsilon \ll M$, we have, from $(p_1+p_2-p_3)^2 = (p_4+k)^2$, neglecting terms of order $(\Delta\epsilon/M)^2$ and $(m/M)^2$,

$$p_2 \cdot (p_1 - p_3) - p_1 \cdot p_3 = M\Delta\epsilon \tag{E1}$$

Writing this in terms of lab frame energies, we have, for high energies,

$$M(\epsilon_1 - \epsilon_3) - \epsilon_1\epsilon_3(1-\cos\theta) = M\Delta\epsilon \tag{E2}$$

For elastic scattering in the lab frame, we have

$$M(\epsilon_1 - \epsilon_3^{el}) - \epsilon_1 \epsilon_3^{el}(1 - \cos\theta) = 0 \tag{E3}$$

Subtracting gives

$$\Delta E \left(1 + \frac{\epsilon_1}{M}(1 - \cos\theta)\right) = \Delta\epsilon \tag{E4}$$

where

$$\Delta E = \epsilon_3^{el} - \epsilon_3 \tag{E5}$$

Thus, in terms of lab frame quantities we have

$$\Delta\epsilon = \eta \Delta E \tag{E6}$$

APPENDIX F: HIGH ENERGY APPROXIMATION FOR $S_{ij}^{(2)}$

In this appendix we give the high energy approximation of the terms $S_{ij}^{(2)}$ defined in (4.15), in which we note in particular that for $i = 1$ or 3 we have $l \cdot t = (\alpha p_i - p_j) \cdot t \approx \alpha p_i \cdot t$. Using transformations of the dilog (Spence) functions [3], p. 389 (B.3),

$$L(z) = -L\left(\frac{1}{z}\right) - \frac{1}{6}\pi^2 - \frac{1}{2}\ln^2(-z) \tag{F1}$$

$$L(z) = -L\left(\frac{z}{z-1}\right) - \frac{1}{2}\ln^2(1-z) \tag{F2}$$

the terms in $S_{ij}^{(2)}$ simplify considerably. We then obtain

$$S_{12}^{(2)} = -\ln^2\left(\frac{2\epsilon_3}{m}\right) - \ln^2 x + \frac{1}{2}\ln^2\left(\frac{x}{\eta}\right) - \frac{1}{6}\pi^2 \\ -L\left(1 - \frac{1}{x\eta}\right) + L\left(1 - \frac{\eta}{x}\right) \tag{F3}$$

$$S_{32}^{(2)} = -\ln^2\left(\frac{2\epsilon_1}{m}\right) - \ln^2(x) + \frac{1}{2}\ln^2(x\eta) - \frac{1}{6}\pi^2 \\ +L\left(1 - \frac{1}{x\eta}\right) - L\left(1 - \frac{\eta}{x}\right) \tag{F4}$$

$$S_{14}^{(2)} = -\ln^2\left(\frac{2\epsilon_3}{m}\right) - \frac{1}{6}\pi^2 \tag{F5}$$

$$S_{34}^{(2)} = -\ln^2\left(\frac{2\epsilon_1}{m}\right) - \frac{1}{6}\pi^2 \tag{F6}$$

$$S_{13}^{(2)} = -\ln^2\left(\frac{2\epsilon_1}{m}\right) - \ln^2\left(\frac{2\epsilon_3}{m}\right) - \frac{1}{3}\pi^2 + \frac{1}{2}\ln^2\left(\cos^2\frac{1}{2}\theta\right) + L\left(\cos^2\frac{1}{2}\theta\right) \tag{F7}$$

$$S_{24}^{(2)} = \frac{1}{2}\ln^2(x) + \frac{1}{2}L\left(1 - \frac{1}{x^2}\right) \tag{F8}$$

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FIG. 1. Feynman diagrams for elastic amplitudes.

FIG. 2. Feynman diagrams for inelastic amplitudes.

FIG. 3. The curves show the contribution of nucleonic size effects (VTX - dashed curve), mathematical refinement (D0 - dotted curve), and the resulting difference (D - solid curve) between the radiative correction given by Mo and Tsai [2] and that given in this paper, δ_{MTj}, as a function of four-momentum transfer for an initial electtron energy of 6 GeV. (VTX, D0, and D are defined in Sec. V.)

FIG. 4. As Fig. 3, but with an initial electron energy of 16 GeV.

TABLE I. Contributions to the radiative correction δ for electron-proton scattering as given in this paper (MTj) and in Mo and Tsai [2] (MoTsai) for three initial electron energies and four-momentum transfers. Values in the rows marked Z^0, Z^1, and Z^2 refer to contributions from terms with these factors in (5.2).

	$\epsilon_1 = 4.4$ GeV $Q^2 = 6$ (GeV/c)^2		$\epsilon_1 = 12$ GeV $Q^2 = 16$ (GeV/c)^2		$\epsilon_1 = 21.5$ GeV $Q^2 = 31.3$ (GeV/c)^2	
	MTj	MoTsai	MTj	MoTsai	MTj	MoTsai
Z^0	−0.2187	−0.2171	−0.2330	−0.2322	−0.2323	−0.2317
Z^1	−0.0569	−0.0506	−0.0517	−0.0479	−0.0625	−0.0571
Z^2	−0.0242	−0.0232	−0.0359	−0.0347	−0.0452	−0.0440
$\delta_{el}^{(1)}$	+0.0068		+0.0116		+0.0185	
δ	−0.2930	−0.2908	−0.3090	−0.3149	−0.3214	−0.3328

TABLE II. Contributions to the radiative correction δ as given in this paper (MTj) and in Mo and Tsai [2] (MoTsai) for several nuclei, with $\epsilon_1 = 4.4$ GeV, $Q^2 = 6$ (GeV/c)^2; other symbols as in Table I.

	^{2}H		^{4}He		^{12}C		^{40}Ca	
	MTj	MoTsai	MTj	MoTsai	MTj	MoTsai	MTj	MoTsai
Z^0	−0.2476	−0.2467	−0.2535	−0.2532	−0.2615	−0.2609	−0.2632	−0.2625
Z^1	−0.0187	−0.0183	−0.0066	−0.0077	−0.0151	−0.0168	−0.0147	−0.0173
Z^2	−0.0094	−0.0088	−0.0077	−0.0071	−0.0156	−0.0145	−0.0188	−0.0178
$\delta_{el}^{(1)}$	+0.0010		+0.0002		+0.0001		+0.0001	
δ	−0.2747	−0.2739	−0.2677	−0.2680	−0.2920	−0.2922	−0.2966	−0.2975

TABLE III. Contributions to the radiative correction δ as given in this paper (MTj) and in Mo and Tsai [2] (MoTsai) for several nuclei, with $\epsilon_1 = 21.5$ GeV, $Q^2 = 31.3$ (GeV/c)^2 ; other symbols as in Table I.

	^{2}H		^{4}He		^{12}C		^{40}Ca	
	MTj	MoTsai	MTj	MoTsai	MTj	MoTsai	MTj	MoTsai
Z^0	−0.2707	−0.2704	−0.2817	−0.2814	−0.2876	−0.2875	−0.2896	−0.2894
Z^1	−0.0182	−0.0189	−0.0149	−0.0166	−0.0134	−0.0164	−0.0123	−0.0173
Z^2	−0.0231	−0.0221	−0.0379	−0.0352	−0.0583	−0.0535	−0.0761	−0.0708
$\delta_{el}^{(1)}$	+0.0045		+0.0026		+0.0006		+0.0001	
δ	−0.3076	−0.3114	−0.3319	−0.3333	−0.3587	−0.3573	−0.3786	−0.3775